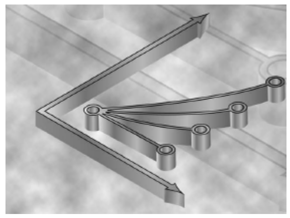
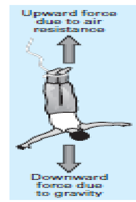
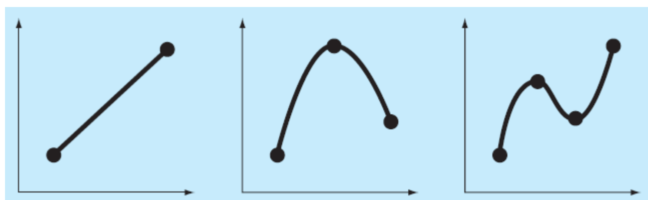
Lecture 01: Introduction to Numerical Methods







What are numerical methods?

- Techniques by which mathematical problems are formulated so that they can be solved with arithmetic and logical operations.
- Because digital computers excel at performing such operations, numerical methods are sometimes referred to as computer mathematics.

Why should you study them?

1. Numerical methods greatly expand the types of problems you can address → capable of handling large systems of equations, nonlinearities, and complicated geometries that are often impossible to solve analytically with standard calculus.
2. Numerical methods allow you to use “canned” software with insight.
3. If you are conversant with numerical methods, and are adept at computer programming, you can design your own programs to solve problems without having to buy or commission expensive software.
4. You will also learn to acknowledge and control the errors of approximation that are part and parcel of large-scale numerical calculations.
5. Numerical methods provide a vehicle for you to reinforce your understanding of mathematics.

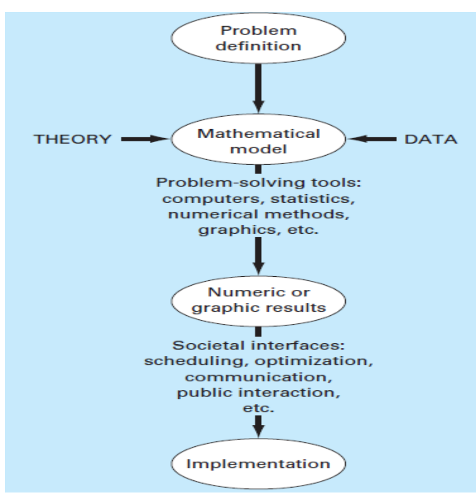
Common Characteristics of Numerical Methods

- Involve large numbers of tedious arithmetic calculations.

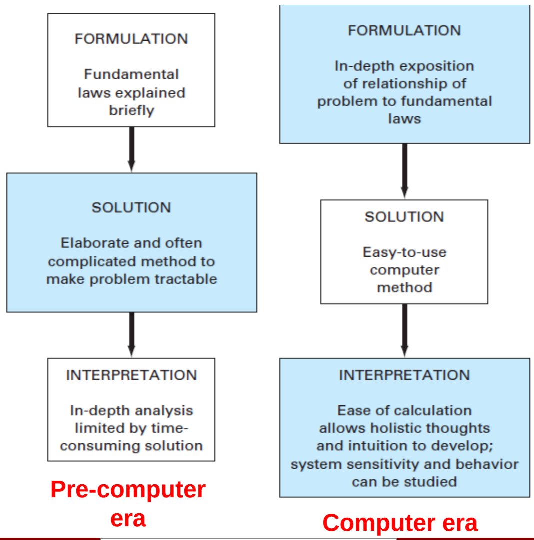
Non-computer Methods

1. Analytical or exact methods (for only a limited class of problems).
2. Graphical solutions.
3. Calculators or slide rules to implement numerical methods manually.

Engineering Problem-solving Process

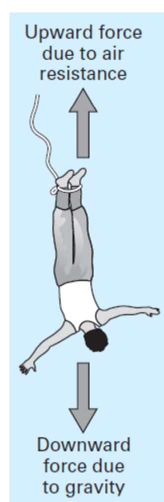


Engineering Problem-solving Process



Free-falling Bungee Jumper

- Suppose that a bungee-jumping company hires you. You are given the task of predicting the velocity of a jumper as a function of time during the free-fall part of the jump. This information will be used as part of a larger analysis to determine the length and required strength of the bungee cord for jumpers of different mass.



Free-falling Bungee Jumper

- Newton's second law → acceleration should be equal to the ratio of the force to the mass.
- Based on your knowledge on of physics and fluid mechanics:

$$\frac{dv}{dt} = g - \frac{c_d}{m} v^2$$

where v = *downward vertical velocity (m/s), t = time (s), g = the acceleration due to* gravity (~= 9.81 m/s^2), c_d = *a lumped drag coefficient (kg/m), and m = the jumper's* mass (kg).

A Simple Mathematical Model

- A mathematical model can be broadly defined as a formulation or equation that expresses the essential features of a physical system or process in mathematical terms.

$$\begin{matrix}\text{Dependent} \\ \text{variable}\end{matrix} = f\left(\begin{matrix}\text{independent} \\ \text{variables}\end{matrix}, \text{parameters}, \begin{matrix}\text{forcing} \\ \text{functions}\end{matrix}\right)$$

- Dependent variable → a characteristic that typically reflects the behavior or state of the system.
- Independent variables → usually dimensions, such as time and space, along which the system's behavior is being determined.
- Parameters → reflective of the system's properties or composition.
- Forcing functions → external influences acting upon it.

Newton's Second Law of Motion as a Mathematical Model

- The time rate of change of momentum of a body is equal to the resultant force acting on it.

$$F = ma$$

$$a = \frac{F}{m}$$

Characteristics of Mathematical Models

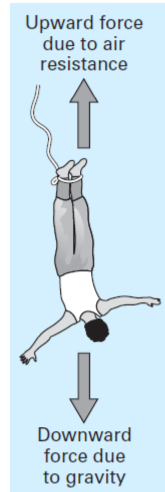
- It describes a natural process or system in mathematical terms.
- It represents an idealization and simplification of reality. That is, the model ignores negligible details of the natural process and focuses on its essential manifestations.
- Finally, it yields reproducible results and, consequently, can be used for predictive purposes.

Newton's second law to determine terminal velocity of a free-falling body near earth's surface

$$a = \frac{F}{m}$$

$$\frac{dv}{dt} = \frac{F}{m}$$

$$F = F_D + F_U \qquad F_D = mg \qquad F_U = -c_d v^2$$

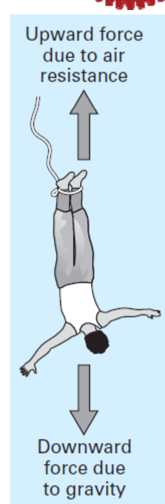


The parameter c_d *accounts for properties of the falling object, such as shape or surface roughness, that affect* air resistance. For the present case, c_d *might be a function of the type of clothing or the* orientation used by the jumper during free fall.

Newton's second law to determine terminal velocity of a free-falling body near earth's surface

$$\frac{dv}{dt} = \frac{F}{m} \qquad F = F_D + F_U \qquad F_D = mg \qquad F_U = -c_d v^2$$

$$\frac{dv}{dt} = g - \frac{c_d}{m} v^2$$



For example, if the jumper is initially at rest (*v = 0 at t = 0*), *calculus* can be used to solve the equation above for:

$$v(t) = \sqrt{\frac{gm}{c_d}} \tanh\left(\sqrt{\frac{gc_d}{m}}\, t\right)$$

$$\tanh x = \frac{e^x - e^{-x}}{e^x + e^{-x}}$$

Analytical Solution to Bungee Jumper Problem

- **Problem Statement**. A bungee jumper with a mass of 68.1 kg leaps from a stationary hot air balloon. Compute velocity for the first 12 s of free fall. Also determine the terminal velocity that will be attained for an infinitely long cord (or alternatively, the jumpmaster is having a particularly bad day!). Use a drag coefficient of 0.25 kg/m.

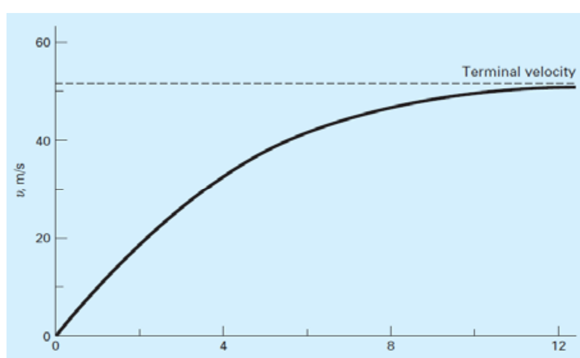
$$v(t) = \sqrt{\frac{gm}{c_d}} \tanh\left(\sqrt{\frac{gc_d}{m}}t\right)$$

$$v(t) = \sqrt{\frac{9.81(68.1)}{0.25}} \tanh\left(\sqrt{\frac{9.81(0.25)}{68.1}}t\right) = 51.6938 \tanh(0.18977t)$$

Analytical Solution to Bungee Jumper Problem

$$v(t) = \sqrt{\frac{9.81(68.1)}{0.25}} \tanh\left(\sqrt{\frac{9.81(0.25)}{68.1}}t\right) = 51.6938 \tanh(0.18977t)$$

t, s	v, m/s
0	0
2	18.7292
4	33.1118
6	42.0762
8	46.9575
10	49.4214
12	50.6175
∞	51.6938



Analytical vs. Numerical Solutions

$$\frac{dv}{dt} = g - \frac{c_d}{m}v^2 \quad \Longrightarrow \quad v(t) = \sqrt{\frac{gm}{c_d}}\tanh\left(\sqrt{\frac{gc_d}{m}}t\right)$$

- This is called an analytical or closed-form solution because it exactly satisfies the original differential equation.
- Unfortunately, there are many mathematical models that cannot be solved exactly.
- In many of these cases, the only alternative is to develop a numerical solution that approximates the exact solution.

Numerical Solution

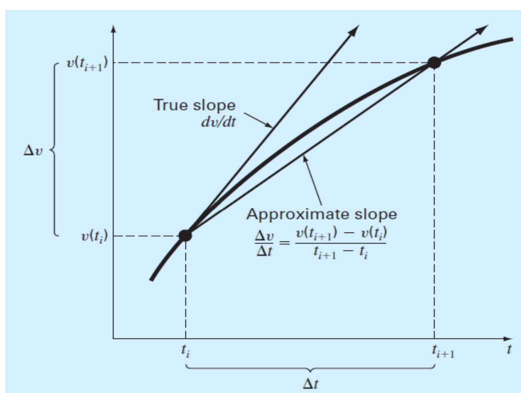
$$\frac{dv}{dt} = g - \frac{c_d}{m}v^2$$

- The time rate of change of velocity can be approximated by

$$\frac{dv}{dt} = \lim_{\Delta t \to 0} \frac{\Delta v}{\Delta t}$$

$$\frac{dv}{dt} \cong \frac{\Delta v}{\Delta t} = \frac{v(t_{i+1}) - v(t_i)}{t_{i+1} - t_i}$$

Finite-difference approximation of the derivative at time t_i



Numerical Solution

$$\frac{dv}{dt} = g - \frac{c_d}{m}v^2$$

$$\frac{dv}{dt} \cong \frac{\Delta v}{\Delta t} = \frac{v(t_{i+1}) - v(t_i)}{t_{i+1} - t_i}$$

$$v(t_{i+1}) = v(t_i) + \left[g - \frac{c_d}{m}v(t_i)^2\right](t_{i+1} - t_i)$$

$$v_{i+1} = v_i + \frac{dv_i}{dt}\Delta t$$

$$\text{New value} = \text{old value} + \text{slope} \times \text{step size}$$

This approach is formally called *Euler's method.*

Numerical Solution to Bungee Jumper Problem

- **Problem Statement**. A bungee jumper with a mass of 68.1 kg leaps from a stationary hot air balloon. Compute velocity for the first 12 s of free fall. Also determine the terminal velocity that will be attained for an infinitely long cord (or alternatively, the jumpmaster is having a particularly bad day!). Use a drag coefficient of 0.25 kg/m. Employ a step size of 2 s for the calculation.

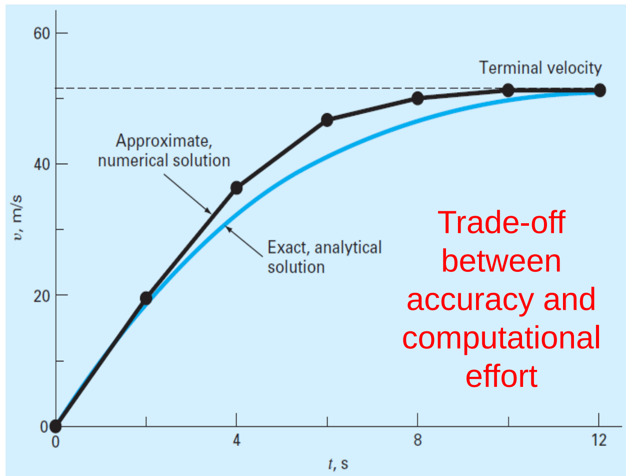
$$v(t_{i+1}) = v(t_i) + \left[g - \frac{c_d}{m}v(t_i)^2\right](t_{i+1} - t_i)$$

$$v = 0 + \left[9.81 - \frac{0.25}{68.1}(0)^2\right] \times 2 = 19.62 \text{ m/s}$$

$$v = 19.62 + \left[9.81 - \frac{0.25}{68.1}(19.62)^2\right] \times 2 = 36.4137 \text{ m/s}$$

Numerical Solution to Bungee Jumper Problem

t, s	*v*, m/s
0	0
2	19.6200
4	36.4137
6	46.2983
8	50.1802
10	51.3123
12	51.6008
∞	51.6938



Conservation Laws in Engineering and Science

$$\text{Change} = \text{increases} - \text{decreases}$$

- This equation embodies one of the most fundamental ways in which conservation laws are used in engineering and science, i.e., to predict changes with respect to time.
- This is referred to as the time-variable (or transient) computation.
- If change is nonexistent:

$$\text{Change} = 0 = \text{increases} - \text{decreases}$$

or

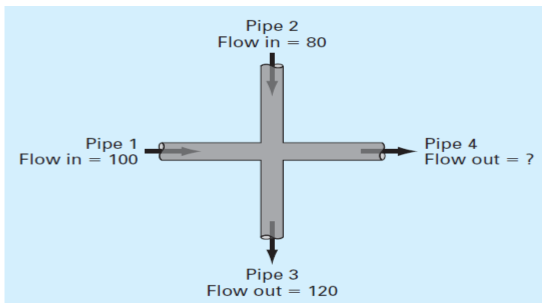
$$\text{Increases} = \text{decreases}$$

- This is referred to as the *steady-state calculation.*

Example: Steady-state calculation

- For steady-state incompressible fluid flow in pipes, the flow into a junction must be balanced by flow going out, as in

Flow in = Flow out



Example: Steady-state calculation

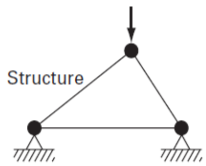
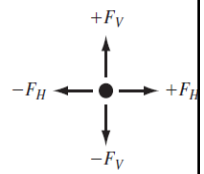
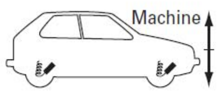
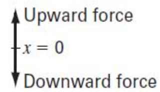
- For the bungee jumper, the steady-state condition would correspond to the case where the net force was zero.

$$mg = c_d v^2$$

$$v = \sqrt{\frac{gm}{c_d}}$$

Terminal velocity

Some models and associated conservation laws

Field	Device	Organizing Principle	Mathematical Expression
Civil engineering		Conservation of momentum	Force balance:  At each node Σ horizontal forces $(F_H) = 0$ Σ vertical forces $(F_V) = 0$
Mechanical engineering		Conservation of momentum	Force balance:  $m\frac{d^2x}{dt^2}$ = downward force − upward force

Case Study

Background. In our model of the free-falling bungee jumper, we assumed that drag depends on the square of velocity (Eq. 1.7). A more detailed representation, which was originally formulated by Lord Rayleigh, can be written as

$$F_d = -\frac{1}{2}\rho v^2 A C_d \vec{v} \tag{1.17}$$

where F_d = the drag force (N), ρ = fluid density (kg/m^3), A = the frontal area of the object on a plane perpendicular to the direction of motion (m^2), C_d = a dimensionless drag coefficient, and $\vec{v}$ = a unit vector indicating the direction of velocity.

This relationship, which assumes turbulent conditions (i.e., a high *Reynolds number*), allows us to express the lumped drag coefficient from Eq. (1.7) in more fundamental terms as

$$C_d = \frac{1}{2}\rho A C_d \tag{1.18}$$

Case Study

Thus, the lumped drag coefficient depends on the object's area, the fluid's density, and a dimensionless drag coefficient. The latter accounts for all the other factors that contribute to air resistance such as the object's "roughness". For example, a jumper wearing a baggy outfit will have a higher C_d than one wearing a sleek jumpsuit.

Note that for cases where velocity is very low, the flow regime around the object will be laminar and the relationship between the drag force and velocity becomes linear. This is referred to as *Stokes drag*.

In developing our bungee jumper model, we assumed that the downward direction was positive. Thus, Eq. (1.7) is an accurate representation of Eq. (1.17), because $\bar{v} = +1$ and the drag force is negative. Hence, drag reduces velocity.

But what happens if the jumper has an upward (i.e., negative) velocity? In this case, $\bar{v} = -1$ and Eq. (1.17) yields a positive drag force. Again, this is physically correct as the positive drag force acts downward against the upward negative velocity.

Unfortunately, for this case, Eq. (1.7) yields a negative drag force because it does not include the unit directional vector. In other words, by squaring the velocity, its sign and hence its direction is lost. Consequently, the model yields the physically unrealistic result that air resistance acts to accelerate an upward velocity!

In this case study, we will modify our model so that it works properly for both downward and upward velocities. We will test the modified model for the same case as Example 1.2, but with an initial value of $v(0) = -40$ m/s. In addition, we will also illustrate how we can extend the numerical analysis to determine the jumper's position.

Case Study: Solution

$$F_U = -c_d v^2 \tag{1.7}$$

Solution. The following simple modification allows the sign to be incorporated into the drag force:

$$F_d = -\frac{1}{2}\rho v|v| A C_d \tag{1.19}$$

or in terms of the lumped drag:

$$F_d = -c_d v|v| \tag{1.20}$$

Thus, the differential equation to be solved is

$$\frac{dv}{dt} = g - \frac{c_d}{m} v|v| \tag{1.21}$$

In order to determine the jumper's position, we recognize that distance travelled, x (m), is related to velocity by

$$\frac{dx}{dt} = -v \tag{1.22}$$

Case Study: Solution

$$\frac{dx}{dt} = -v \qquad (1.22)$$

In contrast to velocity, this formulation assumes that upward distance is positive. In the same fashion as Eq. (1.12), this equation can be integrated numerically with Euler's method:

$$x_{i+1} = x_i - v(t_i)\Delta t \qquad (1.23)$$

Assuming that the jumper's initial position is defined as $x(0) = 0$, and using the parameter values from Examples 1.1 and 1.2, the velocity and distance at $t = 2$ s can be computed as

$$v(2) = -40 + \left[9.81 - \frac{0.25}{68.1}(-40)(40)\right]2 = -8.6326 \text{ m/s}$$

$$x(2) = 0 - (-40)2 = 80 \text{ m}$$

Note that if we had used the incorrect drag formulation, the results would be −32.1274 m/s and 80 m.

Case Study: Solution

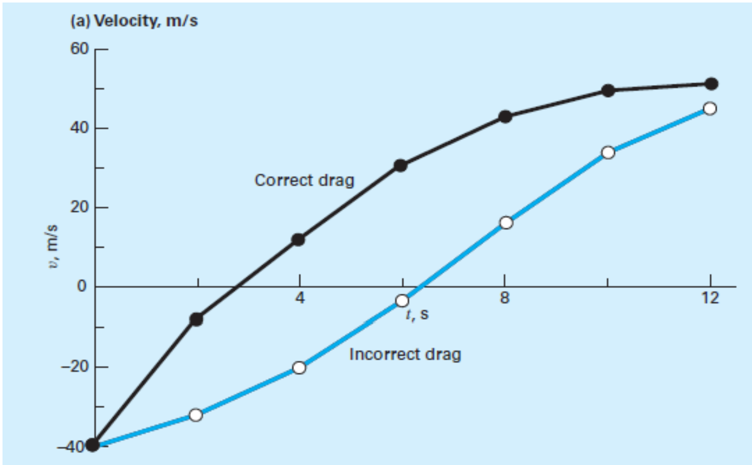
The computation can be repeated for the next interval ($t = 2$ to 4 s):

$$v(4) = -8.6326 + \left[9.81 - \frac{0.25}{68.1}(-8.6326)(8.6326)\right]2 = 11.5346 \text{ m/s}$$

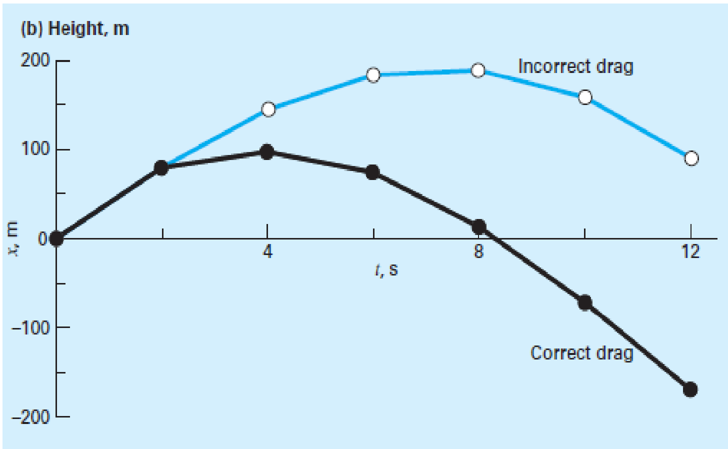
$$x(4) = 80 - (-8.6326)2 = 97.2651 \text{ m}$$

The incorrect drag formulation gives –20.0858 m/s and 144.2549 m.

Case Study: Solution



Case Study: Solution



Assignment-01

- Problems 1.4, 1.7, 1.11, 1.12.