BANGLADESH ARMY INTERNATIONAL UNIVERSITY OF SCIENCE AND TECHNOLOGY (BAIUST), CUMILLA

Term Final Examination, Spring 2024

Department of Computer Science and Engineering (CSE)

Level-1, Term-1

Course Code: CSE 113 **Full Marks:** 150
Course Title: Discrete Mathematics
Credit Hour: 03 **Time:** 2 hr

ANSWER SHEET

Answer to Question 1

Question 1.a: Properties of Relation R

The relation is defined on the set $A = \{1, 2, 3, 4\}$

$$R = \{(1,1),(1,2),(1,3),(1,4),(2,2),(2,3),(2,4),(3,3),(3,4),(4,4)\}$$

i. Reflexive

A relation R on set A is reflexive if $(a, a) \in R$ for every $a \in A$. For $A = \{1, 2, 3, 4\}$, we check for $(1,1),(2,2),(3,3),(4,4)$ in R. Since all these pairs are present in R:

$$(1,1) \in R, (2,2) \in R, (3,3) \in R, (4,4) \in R$$

The relation R is reflexive.

ii. Symmetric

A relation R on set A is symmetric if whenever $(a, b) \in R$, then $(b, a) \in R$. We check for pairs where $a \neq b$:

- $(1,2) \in R$, but $(2,1) \notin R$.
- $(1,3) \in R$, but $(3,1) \notin R$.
- $(1,4) \in R$, but $(4,1) \notin R$.

Since $(1,2) \in R$ but $(2,1) \notin R$, the relation R is not symmetric.

iii. Antisymmetric

A relation R on set A is antisymmetric if whenever $(a, b) \in R$ and $(b, a) \in R$ with $a \neq b$, it must be false (i.e., if $(a, b) \in R$ and $(b, a) \in R$, then $a = b$). We examine all pairs (a, b) where $a \neq b$:

- $(1, 2) \in R$. Is $(2, 1) \in R$? No.
- $(1, 3) \in R$. Is $(3, 1) \in R$? No.
- $\ldots$

For every pair $(a, b) \in R$ where $a \neq b$, the reverse pair (b, a) is not in R. Thus, the condition for antisymmetry is vacuously true for all pairs with $a \neq b$. The relation R is antisymmetric.

iv. Transitive

A relation R on set A is transitive if whenever $(a, b) \in R$ and $(b, c) \in R$, then $(a, c) \in R$. We systematically check all pairs:

- For $a = 1$:
  - $(1, 2) \in R$ and $(2, 3) \in R \implies (1, 3) \in R$. (True)
  - $(1, 2) \in R$ and $(2, 4) \in R \implies (1, 4) \in R$. (True)
  - $(1, 3) \in R$ and $(3, 4) \in R \implies (1, 4) \in R$. (True)
- For $a = 2$:
  - $(2, 3) \in R$ and $(3, 4) \in R \implies (2, 4) \in R$. (True)

All possible compositions (a, b) and (b, c) lead to $(a, c) \in R$. The relation R is transitive.

Property	**Satisfied?**
Reflexive	Yes
Symmetric	No
Antisymmetric	Yes
Transitive	Yes

Question 1.b: Properties of a Graph Relation

The question asks to determine the properties (reflexive, irreflexive, symmetric, antisymmetric, and/or transitive) of a graph with vertices $\{1, 2, 3, 4\}$. Since a graph drawing is not provided for this specific part, and the question structure mirrors 1.a., we assume the graph represents an undirected relation and analyze based on typical graph properties.

If the graph is assumed to be an undirected simple graph on $V = \{1, 2, 3, 4\}$, the relation R on V is defined by $R = \{(a, b) \mid$ there is an edge between a and b or $a = b\}$. The properties would depend entirely on the specific edges in the graph.

Alternatively, if we assume the intention was to analyze the specific graph drawing provided later in the document (Question 2.b), that graph is a directed graph on vertices $\{a, b, c, d\}$, which doesn't match the $\{1, 2, 3, 4\}$ set for 1.b. Therefore, the relation cannot be determined without the correct graph for 1.b.

Question 1.c: Equivalence Relation Proof

Show that the relation $R = \{(a, b) | \ a \equiv b$ (mod m) $\}$ is an equivalence relation on the set of integers $\mathbb{Z}$, where m is an integer with $m > 1$.

To be an equivalence relation, R must satisfy three properties: reflexivity, symmetry, and transitivity. The condition $a \equiv b \pmod{m}$ means that m divides $(a - b)$, written as $m \mid (a - b)$.

i. Reflexive

R is reflexive if $(a, a) \in R$ for all $a \in \mathbb{Z}$.

$$(a, a) \in R \iff a \equiv a \pmod{m}$$

This is true if $m \mid (a - a)$. Since $a - a = 0$, and $m \mid 0$ for any integer $m > 1$, the relation R is reflexive.

ii. Symmetric

R is symmetric if $(a, b) \in R \implies (b, a) \in R$. Assume $(a, b) \in R$.

$$a \equiv b \pmod{m} \implies m \mid (a - b)$$

This means $a - b = km$ for some integer k. Multiplying by -1, we get $-(a - b) = -km$, which simplifies to $b - a = (-k)m$. Since $-k$ is also an integer, $m \mid (b - a)$.

$$m \mid (b - a) \implies b \equiv a \pmod{m} \implies (b, a) \in R$$

The relation R is symmetric.

iii. Transitive

R is transitive if $(a, b) \in R$ and $(b, c) \in R \implies (a, c) \in R$. Assume $(a, b) \in R$ and $(b, c) \in R$.

$$(a, b) \in R \implies a \equiv b \pmod{m} \implies m \mid (a - b) \implies a - b = k_1 m \quad (\text{for integer } k_1)$$
$$(b, c) \in R \implies b \equiv c \pmod{m} \implies m \mid (b - c) \implies b - c = k_2 m \quad (\text{for integer } k_2)$$

Adding the two equations:

$$(a-b)+(b-c)=k_1m+k_2m$$

$$a-c=(k_1+k_2)m$$

Since k_1+k_2 is an integer, $m \mid (a-c)$.

$$m \mid (a-c) \implies a \equiv c \pmod{m} \implies (a,c) \in R$$

The relation R is transitive.

Since R is reflexive, symmetric, and transitive, it is an equivalence relation.

Answer to Question 2

Question 2.a: Number of Edges in a Regular Graph

Question: How many edges are there in a graph with 15 vertices each of degree eight?

Solution using the Handshaking Theorem

The Handshaking Theorem states that the sum of the degrees of the vertices is equal to twice the number of edges:

$$\sum_{v \in V} \text{deg}(v) = 2|E|$$

Given:

- Number of vertices, $|V| = n = 15$
- Degree of each vertex, $\text{deg}(v) = 8$

The sum of the degrees is:

$$\sum_{v \in V} \text{deg}(v) = 15 \times 8 = 120$$

Solving for the number of edges, $|E|$:

$$2|E| = 120$$

$$|E| = \frac{120}{2}$$

$$|E| = 60$$

The graph has 60 edges.

Question 2.b: In-degree and Out-degree in a Directed Graph

Question: Find the in-degree and out-degree of each vertex in the given graph with directed edges.

Solution by Inspection

The graph has vertices $V = \{a, b, c, d\}$ (referencing the image on page 2).

1. **Out-degree** (Number of edges leaving the vertex):
   - out-deg(a): Edges to b (3), to c (2), to d (1). $\implies 3 + 2 + 1 = 6$.
   - out-deg(b): Edge to c (1), edge to d (1). $\implies 1 + 1 = 2$.
   - out-deg(c): Edge to b (1). $\implies 1$.
   - out-deg(d): Edge to a (1), edges to c (3). $\implies 1 + 3 = 4$.
2. **In-degree** (Number of edges entering the vertex):
   - in-deg(a): Edge from d (1). $\implies 1$.
   - in-deg(b): Edges from a (3), edge from c (1). $\implies 3 + 1 = 4$.
   - in-deg(c): Edges from a (2), edge from b (1), edges from d (3). $\implies 2+1+3 = 6$.
   - in-deg(d): Edge from a (1), edge from b (1). $\implies 1 + 1 = 2$.

Summary Table

Vertex	In-degree	Out-degree
a	1	6
b	4	2
c	6	1
d	2	4

Question 2.c: Pigeonhole Principle

Question: Show that if 30 dictionaries in a library contain a total of 61327 pages, then one of the dictionaries must have 2045 pages.

Solution using the Generalized Pigeonhole Principle

The Generalized Pigeonhole Principle states that if N items are placed into k boxes, then there is at least one box containing at least $\lceil N/k \rceil$ items.

- N (Total pages/Items) $= 61327$
- k (Number of dictionaries/Boxes) $= 30$

The minimum number of pages that must be in at least one dictionary is:

$$\text{Min Pages} = \left\lceil \frac{N}{k} \right\rceil = \left\lceil \frac{61327}{30} \right\rceil$$

Calculate the value:

$$\frac{61327}{30} = 2044.233\ldots$$

Applying the ceiling function:

$$\lceil 2044.233\ldots \rceil = 2045$$

Conclusion: By the Generalized Pigeonhole Principle, at least one of the 30 dictionaries must contain **2045 pages**. If every dictionary had 2044 pages or less, the maximum total number of pages would be $30 \times 2044 = 61320$, which is less than the given total of 61327 pages.

Answer to Question 3

Question 3.a: Strongly Connected Component and Weakly Connected Component

Question: Explain Strongly connected component and Weakly connected component with an example.

Strongly Connected Component (SCC)

A Strongly Connected Component (SCC) is a maximal subgraph of a directed graph where, for every pair of vertices (u, v) in the subgraph, there is a directed path from u to v AND a directed path from v to u. In simpler terms, every vertex in an SCC is reachable from every other vertex in that same component by following the direction of the arrows.

Example: Consider a directed graph with vertices $V = \{1, 2, 3, 4\}$ and edges $E = \{(1, 2), (2, 1), (2, 3), (3, 4), (4, 3)\}$.

- There is a path $1 \to 2$ and $2 \to 1$. Thus, $\{1, 2\}$ is an SCC.
- There is a path $3 \to 4$ and $4 \to 3$. Thus, $\{3, 4\}$ is an SCC.

The graph has two SCCs: $\{\mathbf{1}, \mathbf{2}\}$ and $\{\mathbf{3}, \mathbf{4}\}$. The vertices 1 and 3, for instance, are not strongly connected because while there is a path $1 \to 2 \to 3$, there is no directed path back from 3 to 1.

Weakly Connected Component (WCC)

A Weakly Connected Component (WCC) is a maximal subgraph of a directed graph where the underlying undirected graph is connected. The underlying undirected graph is the graph formed by replacing all directed edges with undirected edges (ignoring the direction). In a WCC, every vertex is reachable from every other vertex if you are allowed to ignore the direction of the arrows.

Example: Using the same directed graph as above: $E = \{(1,2),(2,1),(2,3),(3,4),(4,3)\}$.

- If we ignore the direction, we have undirected paths between all vertices: $1 \leftrightarrow 2 \leftrightarrow 3 \leftrightarrow 4$.
- All vertices $\{1,2,3,4\}$ are connected in the underlying undirected graph.

The entire graph $\{\mathbf{1},\mathbf{2},\mathbf{3},\mathbf{4}\}$ is a single Weakly Connected Component.

Question 3.b: Graph Isomorphism

Question: Are these graphs $G1$ and $G2$ isomorphic? If yes/not explain why.

Analysis of Graph $G1$

$G1$ (Left graph) has vertices $V_1 = \{A,B,D,E,C\}$ (5 vertices). Edges $E_1 = \{(A,B),(A,D),(B,D),(B,$
The number of edges is $|E_1| = 5$.

Degrees of vertices in $G1$:

- deg$(A) = 2$ (A to B, A to D)
- deg$(B) = 3$ (B to A, B to D, B to E)
- deg$(C) = 1$ (C to E)
- deg$(D) = 2$ (D to A, D to B)
- deg$(E) = 2$ (E to B, E to C)

Degree sequence of $G1$: $(\mathbf{3},\mathbf{2},\mathbf{2},\mathbf{2},\mathbf{1})$.

Analysis of Graph $G2$

$G2$ (Right graph) has vertices $V_2 = \{A,C,D,B,E\}$ (5 vertices)[cite: 53, 54, 57, 59, 61, 62]. Edges $E_2 = \{(A,B),(A,C),(B,C),(C,E),(D,E)\}$. The number of edges is $|E_2| = 5$.

Degrees of vertices in $G2$:

- deg$(A) = 2$ (A to B, A to C)
- deg$(B) = 2$ (B to A, B to C)

- deg$(C) = 3$ (C to A, C to B, C to E)
- deg$(D) = 1$ (D to E)
- deg$(E) = 2$ (E to C, E to D)

Degree sequence of $G2$: $(\mathbf{3}, \mathbf{2}, \mathbf{2}, \mathbf{2}, \mathbf{1})$.

Conclusion on Isomorphism

Two graphs are **isomorphic** if they have the same number of vertices, the same number of edges, and the same degree sequence. If they are isomorphic, there must be a one-to-one correspondence (mapping) between their vertices that preserves adjacency.

- $|V_1| = 5$ and $|V_2| = 5$. (Match)
- $|E_1| = 5$ and $|E_2| = 5$. (Match)
- Degree sequence of $G1$ is $(3, 2, 2, 2, 1)$ and $G2$ is $(3, 2, 2, 2, 1)$. (Match)

Since all basic invariants match, we attempt to find the isomorphism mapping $f : V_1 \to V_2$:

- Map the unique degree 3 vertex: $f(B) = C$
- Map the unique degree 1 vertex: $f(C) = D$
- C in $G1$ is connected only to E. D in $G2$ is connected only to E. Therefore, $f(E) = E$.

Now we check the remaining vertices $\{A, D\}$ in $G1$ and $\{A, B\}$ in $G2$:

- A in $G1$ is adjacent to B (deg 3) and D (deg 2).
- D in $G1$ is adjacent to A (deg 2) and B (deg 3).
- A in $G2$ is adjacent to C (deg 3) and B (deg 2).
- B in $G2$ is adjacent to C (deg 3) and A (deg 2).

We can set the remaining mapping as $f(A) = A$ and $f(D) = B$.

Let's verify edges with this mapping f:

- $(A, B) \in E_1 \implies (f(A), f(B)) = (A, C) \in E_2$. (True)
- $(A, D) \in E_1 \implies (f(A), f(D)) = (A, B) \in E_2$. (True)
- $(B, D) \in E_1 \implies (f(B), f(D)) = (C, B) \in E_2$. (True)
- $(B, E) \in E_1 \implies (f(B), f(E)) = (C, E) \in E_2$. (True)
- $(C, E) \in E_1 \implies (f(C), f(E)) = (D, E) \in E_2$. (True)

Since a successful adjacency-preserving mapping (isomorphism) exists, the graphs **are isomorphic**.

Question 3.c: Set Operations on Relations

Question: Let $A = \{1, 2, 3\}$ and $B = \{1, 2, 3, 4\}$. The relations $R1 = \{(1, 2), (2, 3), (3, 4)\}$ and $R2 = \{(1, 1), (1, 2), (2, 1), (2, 2), (2, 3), (3, 1), (3, 2), (3, 3), (3, 4)\}$. Find:

1. $R1 \cup R2$
2. $R1 \cap R2$
3. $R1 - R2$
4. $R2 - R1$

i. $R1 \cup R2$ **(Union)**

The union includes all pairs present in $R1$, $R2$, or both.

$$R1 \cup R2 = R2 \cup \{(1, 2), (2, 3), (3, 4)\}$$

$$R1 \cup R2 = \{(1, 1), (1, 2), (2, 1), (2, 2), (2, 3), (3, 1), (3, 2), (3, 3), (3, 4)\}$$

Note: Since $R1 \subset R2$ (every element in $R1$ is also in $R2$), their union is simply $R2$.

ii. $R1 \cap R2$ **(Intersection)**

The intersection includes only the pairs present in *both* $R1$ and $R2$.

- $(1, 2)$: In $R1$ and $R2$. (Included)
- $(2, 3)$: In $R1$ and $R2$. (Included)
- $(3, 4)$: In $R1$ and $R2$. (Included)

$$R1 \cap R2 = \{(1, 2), (2, 3), (3, 4)\}$$

Note: Since $R1 \subset R2$, their intersection is simply $R1$.

iii. $R1 - R2$ **(Difference)**

The difference $R1 - R2$ includes pairs that are in $R1$ but **not** in $R2$.

- $(1, 2)$: In $R1$ and in $R2$. (Excluded)
- $(2, 3)$: In $R1$ and in $R2$. (Excluded)
- $(3, 4)$: In $R1$ and in $R2$. (Excluded)

Since all elements of $R1$ are in $R2$, the resulting set is the empty set $\emptyset$.

$$R1 - R2 = \emptyset$$

iv. $R2 - R1$ **(Difference)**

The difference $R2 - R1$ includes pairs that are in $R2$ but **not** in $R1$. We remove all elements of $R1$ from $R2$. $R1 = \{(1,2),(2,3),(3,4)\}$.

$$R2 - R1 = \{(1,1),(2,1),(2,2),(3,1),(3,2),(3,3)\}$$

Answer to Question 4

Question 4.a: Directed Multigraph and Chromatic Number

Question: Explain Directed Multigraph with an example. Find Chromatic number from the following graph.

Directed Multigraph Explanation

A Directed Multigraph is a type of directed graph where it is permitted to have multiple directed edges (also called parallel edges) between the same pair of vertices. It can also include loops, which are edges that connect a vertex back to itself.

Example: Consider two vertices A and B.

- If there are two arrows pointing from A to B, that constitutes multiple edges.
- If there is an arrow that starts at A and ends at A, that is a loop.

These multiple edges and loops distinguish a directed multigraph from a simple directed graph, which only allows at most one unique edge between any two ordered vertices.

Chromatic Number Determination

The Chromatic Number ($\chi(G)$) is the minimum number of colors required to color the vertices of an undirected graph such that no two adjacent vertices have the same color.

Analysis of the Given Graph (A-H on page 3):

1. **Identify the largest clique ($\omega(G)$):** A clique is a subset of vertices where every vertex is connected to every other vertex. We observe that vertices $\mathbf{A}, \mathbf{B}, \mathbf{C}$ form a triangle, since edges $(A, B), (A, C)$, and (B, C) all exist. A triangle is a clique of size 3, so $\omega(G) \geq 3$. This means we need at least 3 colors.

2. **Attempt Coloring with 3 Colors (c_1, c_2, c_3):**

- **Color A, B, C:**
  - A: c_1
  - B: c_2
  - C: c_3

- **Color the remaining vertices (D, E, F, G, H):**
  - D: Adjacent to C (c_3). Assign $D = c_1$.
  - E: Adjacent to C (c_3) and D (c_1). Must be a different color. Assign $E = c_2$.
  - F: Adjacent to D (c_1) and E (c_2). Must be a different color. Assign $F = c_3$.
  - G: Adjacent to E (c_2) and F (c_3). Assign $G = c_1$.
  - H: Adjacent to F (c_3) and G (c_1). Assign $H = c_2$.

Since the graph can be properly colored with 3 colors, and we know $\chi(G) \geq 3$, the Chromatic Number is $\chi(\mathbf{G}) = \mathbf{3}$.

Question 4.b: Euler Path and Euler Circuit

Question: Which of the following graphs has a Euler path and Euler circuit?

Euler Path/Circuit Rules (for Connected Graphs)

- Euler Circuit exists if and only if every vertex has an even degree.
- Euler Path exists (but not a circuit) if and only if there are exactly two vertices of odd degree.

Analysis by Vertex Degree (referencing the three graphs on page 3)

1. **Graph** $G1$ **(Top Left, 5 vertices):**
   - Degrees: $A(2), B(3), C(2), D(2), E(3)$.
   - Odd degrees: Two (B, E).
   - Conclusion: Has an Euler Path (Yes), Euler Circuit (No).
2. **Graph** $G2$ **(Middle, 5 vertices):**
   - Degrees: $A(2), B(3), C(3), D(3), E(3)$.
   - Odd degrees: Four (B, C, D, E).
   - Conclusion: Has an Euler Path (No), Euler Circuit (No).
3. **Graph** $G3$ **(Top Right, 5 vertices):**
   - Degrees: $A(3), B(3), C(3), D(4), E(3)$.
   - Odd degrees: Four (A, B, C, E).
   - Conclusion: Has an Euler Path (No), Euler Circuit (No).

Final Answer for 4.b: Only **G1** has an Euler Path. None of the graphs have an Euler Circuit.

Question 4.c: Hamilton Path and Hamilton Circuit

Question: Which of these three figures has a Hamilton path and Hamilton circuit?

Hamilton Path/Circuit Definitions

- Hamilton Path: A simple path that visits every vertex exactly once.
- Hamilton Circuit: A simple circuit that visits every vertex exactly once, except for the starting vertex which is visited at the end.

(Note: If a Hamilton Circuit exists, a Hamilton Path also exists.)

Analysis by finding Path/Circuit (referencing the three graphs on page 3)

1. **Graph $G1$ (Top Left):**
   - **Hamilton Circuit:** $A \to B \to C \to D \to E \to A$. (All 5 vertices visited).
   - Conclusion: Has both a **Hamilton Path** and a **Hamilton Circuit**.
2. **Graph $G2$ (Middle):**
   - **Hamilton Circuit:** $A \to B \to D \to C \to E \to A$. (All 5 vertices visited).
   - Conclusion: Has both a Hamilton Path and a Hamilton Circuit.
3. **Graph $G3$ (Top Right):**
   - **Hamilton Circuit:** $A \to B \to C \to E \to D \to A$. (All 5 vertices visited).
   - Conclusion: Has both a **Hamilton Path** and a Hamilton Circuit.

Final Answer for 4.c: All three graphs, **G1**, **G2**, and **G3**, have both a Hamilton Path and a Hamilton Circuit.

Answer to Question 5

Question 5.a: Shortest Distance from Source Vertex 1 (Dijkstra's Algorithm)

Question: Find the shortest distance from the source vertex 1 to ... (The rest of the question and the accompanying graph are missing. We assume the question asks for the shortest distance to all other vertices using Dijkstra's Algorithm).

Method: Dijkstra's Algorithm

Dijkstra's Algorithm finds the shortest path from a single source vertex to all other vertices in a graph with non-negative edge weights.

1. Initialize the distance to the source vertex (1) as 0, and the distance to all other vertices as infinity (∞).
2. Mark the current vertex as $v = 1$.
3. For the current vertex, update the distances of its unvisited neighbors if a shorter path is found:

$$\text{Distance}(w) = \min(\text{Distance}(w), \text{Distance}(v) + \text{weight}(v, w))$$

4. Mark the current vertex as visited.
5. Select the unvisited vertex with the smallest known distance and set it as the new current vertex (v).
6. Repeat steps 3–5 until all vertices are visited.

Hypothetical Example Demonstration

Since the actual graph is not provided, we demonstrate the process using a hypothetical graph with vertices $V = \{1, 2, 3, 4\}$ and non-negative weights.

Hypothetical Adjacency/Weight Table (Placeholder Data)

To	1	2	3	4
From 1	0	3	1	∞
From 2	∞	0	1	5
From 3	∞	∞	0	2
From 4	∞	∞	∞	0

Dijkstra's Table (Shortest Distance from Source $s = 1$):

Step	Selected Vertex	1 (Fixed)	2	3	4	Visited Set
0	Initial	0	∞	∞	∞	$\emptyset$
1	1	0	3	1	∞	$\{1\}$
2	3 (Dist=1)	0	min$(3, 1 + \infty)$	1	min$(\infty, 1 + 2)$	$\{1, 3\}$
3	2 (Dist=3)	0	3	1	min$(3, 3 + 5)$	$\{1, 3, 2\}$
4	4 (Dist=3)	0	3	1	3	$\{1, 3, 2, 4\}$

Final Result (Based on Hypothetical Graph)

The shortest distances from the source vertex 1 are:

- Distance$(1) = 0$
- Distance$(2) = 3$ (Path $1 \to 2$)
- Distance$(3) = 1$ (Path $1 \to 3$)
- Distance$(4) = 3$ (Path $1 \to 3 \to 4$)

Note: Please replace the hypothetical data and steps above with the actual graph provided in the exam to obtain the correct solution.

Question 5.b: Graph Representation Techniques and Handshaking Theorem

Question: Describe graph representation techniques. Define the Handshaking Theorem in an undirected graph.

i. Graph Representation Techniques

1. Adjacency Matrix

- An $N \times N$ matrix, where N is the number of vertices.
- The entry $A[i][j]$ is 1 if an edge exists from vertex i to vertex j (0 otherwise). For a weighted graph, $A[i][j]$ stores the edge weight.
- Advantage: Fast $O(1)$ lookup to check if an edge exists.
- Disadvantage: Requires $O(N^2)$ space, inefficient for graphs with few edges (sparse graphs).

2. Adjacency List

- An array or list where each index i corresponds to a vertex and stores a list of its neighbors.
- Advantage: Space efficient, requiring $O(|V|+|E|)$ space, making it ideal for sparse graphs.
- Disadvantage: Slower to check for edge existence (must search the list).

ii. Handshaking Theorem Definition

The Handshaking Theorem applies to undirected graphs.

Definition The theorem states that the sum of the degrees of all vertices in an undirected graph is equal to exactly twice the number of edges in that graph.

$$\sum_{v \in V} \deg(v) = 2|E|$$

This is because every edge connects two vertices, contributing 1 to the degree of each, and therefore 2 to the total sum of degrees.

iii. Application to the Directed Graph (Extended Principle)

While the original theorem is for undirected graphs, the equivalent principle for a directed graph is that the sum of the in-degrees must equal the sum of the out-degrees, and both sums must equal the total number of edges.

Directed Graph Handshaking Principle

$$\sum_{v \in V} \text{in-deg}(v) = \sum_{v \in V} \text{out-deg}(v) = |E|$$

The graph has 6 vertices and 8 edges: $E = \{(1,2), (1,3), (2,3), (2,4), (3,5), (5,4), (5,6), (4,6)\}$.

Vertex (v)	1	2	3	4	5	6	**Sum**
out-deg(v)	2	2	1	1	2	0	**8**
in-deg(v)	0	1	2	2	1	2	**8**

Since $\sum \text{in-deg}(v) = 8$ and $\sum \text{out-deg}(v) = 8$, the Directed Handshaking Principle is satisfied, and the total number of edges $|E|$ is 8.

Answer to Question 6

Question 6.a: Tree Traversals

Question: Find preorder, inorder and postorder traversal from the following graph (tree structure).

Tree Structure Analysis: The graph is a tree rooted at **A**. The structure is:

- Root: **A**
- Left Subtree: **B** (parent of C), **C** (parent of D).
- Right Subtree: **E** (parent of F and K).
- Subtree rooted at F: **F** (parent of I and G).
- Subtree rooted at G: **G** (parent of J and H).
- Subtree rooted at K: **K** (parent of L).

i. Preorder Traversal (Root, Left, Right)

Visit the root first, then traverse the left subtree, then traverse the right subtree.

$$\mathbf{A}, \mathbf{B}, C, D, \mathbf{E}, F, I, G, J, H, K, L$$

ii. Inorder Traversal (Left, Root, Right)

Traverse the left subtree, visit the root, then traverse the right subtree.

$$D, C, B, \mathbf{A}, I, F, J, G, H, E, L, K$$

iii. Postorder Traversal (Left, Right, Root)

Traverse the left subtree, traverse the right subtree, then visit the root last.

$$D, C, B, I, J, H, G, F, L, K, E, \mathbf{A}$$

Question 6.b: Prim's vs. Kruskal's and Kruskal's MST

Question: Differentiate between Prim's and Kruskal's algorithms. Find MST using Kruskal's Algorithm from the following graph.

i. Differentiation between Prim's and Kruskal's Algorithms

Both Prim's and Kruskal's algorithms are greedy methods used to find a **Minimal Spanning Tree (MST)** in a weighted graph.

Feature	Prim's Algorithm	Kruskal's Algorithm
Core Strategy	Grows the MST directly from a chosen starting vertex.	Considers all edges and adds them to the forest.
Focus	Vertices. It focuses on finding the minimum edge connecting a **visited vertex** to an unvisited vertex.	Edges. It focuses on adding the minimum weight edge from the entire set that does not form a **cycle**.
Intermediate State	Produces a single tree at every step.	Produces a forest (a set of disconnected trees) until they merge into a single MST.

ii. Minimal Spanning Tree using Kruskal's Algorithm

Kruskal's Algorithm selects edges in increasing order of weight, adding an edge to the MST only if it does not create a cycle. The graph has $V = 9$ vertices (A, B, C, D, E, F, G, H, I), so the MST must contain $|V| - 1 = 8$ edges.

1. Sorted Edges (Ascending Order of Weight):

1. (H, G): 1
2. (C, I): 2
3. (G, F): 2
4. (A, B): 4
5. (C, F): 4
6. (I, G): 6
7. (C, D): 7
8. (I, H): 7
9. (A, H): 8
10. (B, C): 8
11. (D, E): 9
12. (E, F): 10
13. (B, H): 11
14. (D, F): 14

2. Selection Process:

- Select (H, G) w=1. (MST Edges: 1)
- Select (C, I) w=2. (MST Edges: 2)
- Select (G, F) w=2. (MST Edges: 3)
- Select (A, B) w=4. (MST Edges: 4)
- Select (C, F) w=4. Does not form a cycle. (MST Edges: 5)
- Reject (I, G) w=6. Forms a cycle with edges (C, I), (C, F), (G, F).
- Select (C, D) w=7. (MST Edges: 6)
- Reject (I, H) w=7. Forms a cycle with edges (H, G), (G, F), (C, F), (C, I).
- Select (A, H) w=8. Does not form a cycle. (MST Edges: 7)
- Reject (B, C) w=8. Forms a cycle with edges (A, B), (A, H), (I, C), $\ldots$
- Select (D, E) w=9. This connects vertex E to the existing MST structure. (MST Edges: 8)

3. Final Minimal Spanning Tree (MST) The **8** edges that form the MST are:

$$\text{MST Edges} = \{(H, G), (C, I), (G, F), (A, B), (C, F), (C, D), (A, H), (D, E)\}$$

The **Total Weight** of the MST is the sum of these edge weights:

$$\text{Total Weight} = 1 + 2 + 2 + 4 + 4 + 7 + 8 + 9 = \mathbf{37}$$