Analysis of Variance

Chapter 12

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Learning Objectives

LO1 List the characteristics of the *F* distribution and locate values in an F table.

LO2 Perform a test of hypothesis to determine whether the variances of two populations are equal.

LO3 Describe the ANOVA approach for testing difference in sample means.

LO4 Organize data into ANOVA tables for analysis.

LO5 Conduct a test of hypothesis among three or more treatment means and describe the results.

LO6 Develop confidence intervals for the difference in treatment means and interpret the results.

LO7 Carry out a test of hypothesis among treatment means using a blocking variable and understand the results.

LO8 Perform a two-way ANOVA with interaction and describe the results.

LO1 List the characteristics of the *F* distribution and locate values in an F table.

The *F* Distribution

Uses of the F Distribution

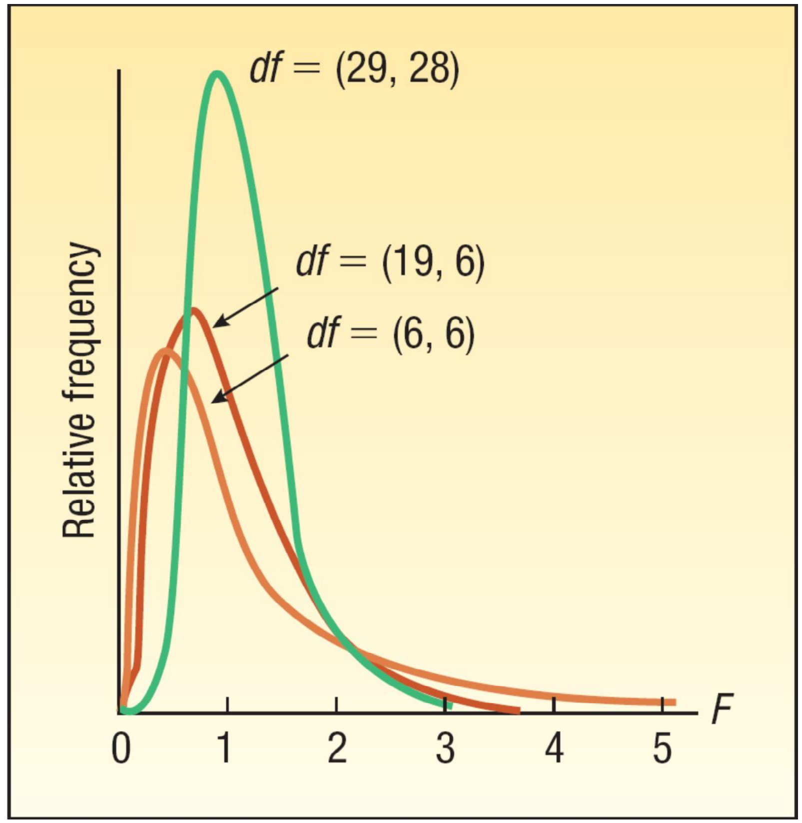
- test whether two samples are from populations having equal variances
- to compare several population means simultaneously. The simultaneous comparison of several population means is called analysis of variance(ANOVA).

Assumption:

In both of the uses above, the populations must follow a normal distribution, and the data must be at least interval-scale.

Characteristics of the F Distribution

1. There is a "family" of *F* Distributions. A particular member of the family is determined by two parameters: the degrees of freedom in the numerator and the degrees of freedom in the denominator.
2. The *F* distribution is continuous
3. *F* cannot be negative.
4. The *F* distribution is positively skewed.
5. It is asymptotic. As $F \rightarrow \infty$ the curve approaches the *X*-axis but never touches it.



LO2 Perform a test of hypothesis to determine whether the variances of two populations are equal.

Comparing Two Population Variances

The *F* distribution is used to test the hypothesis that the variance of one normal population equals the variance of another normal population.

Examples:

- Two Barth shearing machines are set to produce steel bars of the same length. The bars, therefore, should have the same mean length. We want to ensure that in addition to having the same mean length they also have similar variation.
- The mean rate of return on two types of common stock may be the same, but there may be more variation in the rate of return in one than the other. A sample of 10 technology and 10 utility stocks shows the same mean rate of return, but there is likely more variation in the Internet stocks.
- A study by the marketing department for a large newspaper found that men and women spent about the same amount of time per day reading the paper. However, the same report indicated there was nearly twice as much variation in time spent per day among the men than the women.

$$H_0: \sigma_1^2 = \sigma_2^2$$
$$H_1: \sigma_1^2 \neq \sigma_2^2$$

TEST STATISTIC FOR COMPARING TWO VARIANCES

$$F = \frac{s_1^2}{s_2^2}$$

To conduct the test, we select a random sample of n_1 observations from one population, and a random sample of n_2 observations from the second population. The test statistic is defined as follows.

Test for Equal Variances - Example

Lammers Limos offers limousine service from the city hall in Toledo, Ohio, to Metro Airport in Detroit. Sean Lammers, president of the company, is considering two routes. One is via U.S. 25 and the other via I-75. He wants to study the time it takes to drive to the airport using each route and then compare the results. He collected the following sample data, which is reported in minutes.

Using the .10 significance level, **is there a difference in the variation** in the driving times for the two routes?

U.S. Route 25	Interstate 75
52	59
67	60
56	61
45	51
70	56
54	63
64	57
	65

Step 1: The hypotheses are:

$H_0: \sigma_1^2 = \sigma_2^2$

$H_1: \sigma_1^2 \neq \sigma_2^2$

Step 2: The significance level is .05.

Step 3: The test statistic is the F distribution.

Step 4: State the decision rule.

Reject H_0 if $F > F_{\alpha/2,v1,v2}$

$F > F_{.10/2,7-1,8-1}$

$F > F_{.05,6,7}$

$F > 3.87$

TABLE 12–1 Critical Values of the *F* Distribution, α = .05

Degrees of Freedom for Denominator	Degrees of Freedom for Numerator			
	5	6	7	8
1	230	234	237	239
2	19.3	19.3	19.4	19.4
3	9.01	8.94	8.89	8.85
4	6.26	6.16	6.09	6.04
5	5.05	4.95	4.88	4.82
6	4.39	4.28	4.21	4.15
7	3.97	3.87	3.79	3.73
8	3.69	3.58	3.50	3.44
9	3.48	3.37	3.29	3.23
10	3.33	3.22	3.14	3.07

Test for Equal Variances - Example

Step 5: Compute the value of F and make a decision

U.S. Route 25

$$\bar{X} = \frac{\Sigma X}{n} = \frac{408}{7} = 58.29 \qquad s = \sqrt{\frac{\Sigma(X - \bar{X})^2}{n - 1}} = \sqrt{\frac{485.43}{7 - 1}} = 8.9947$$

Interstate 75

$$\bar{X} = \frac{\Sigma X}{n} = \frac{472}{8} = 59.00 \qquad s = \sqrt{\frac{\Sigma(X - \bar{X})^2}{n - 1}} = \sqrt{\frac{134}{8 - 1}} = 4.3753$$

$$F = \frac{s_1^2}{s_2^2} = \frac{(8.9947)^2}{(4.3753)^2} = 4.23$$

The decision is to **reject the null hypothesis**, because the computed *F* value (4.23) is larger than the critical value (3.87).

We conclude that **there is a difference in the variation** of the travel times along the two routes.

LO3 Describe the ANOVA approach for testing difference in sample means.

Comparing Means of Two or More Populations

The *F* distribution is also used for testing whether two or more sample means came from the same or equal populations.

Assumptions:

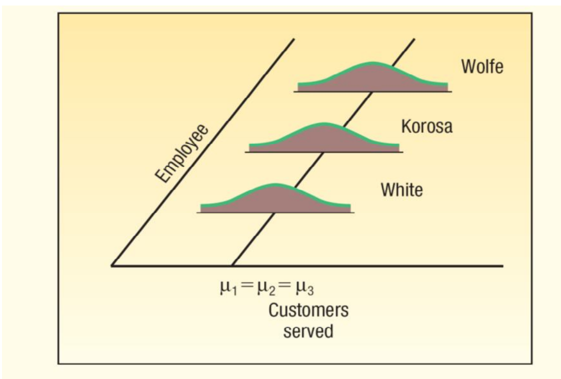
- The sampled populations follow the normal distribution.
- The populations have equal standard deviations.
- The samples are randomly selected and are independent.

The Null Hypothesis is that the population means are the same. The Alternative Hypothesis is that at least one of the means is different.

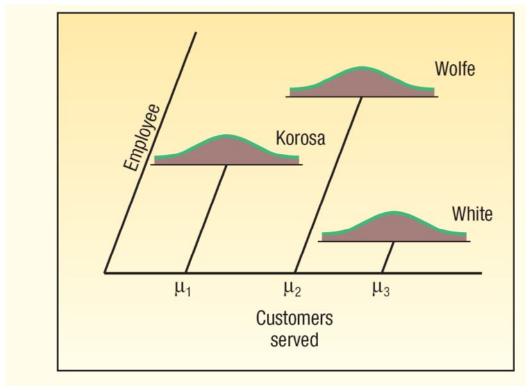
H_0: $\mu_1 = \mu_2 = \ldots = \mu_k$

H_1: The means are not all equal

Reject H_0 if $F > F_{\alpha,k-1,n-k}$



Case Where Treatment Means Are the Same



Case Where Treatment Means Are Different

LO5 Conduct a test of hypothesis among three or more treatment means and describe the results.

Comparing Means of Two or More Populations – Example

EXAMPLE

Recently a group of four major carriers joined in hiring Brunner Marketing Research, Inc., to survey recent passengers regarding their level of satisfaction with a recent flight. The survey included questions on ticketing, boarding, in-flight service, baggage handling, pilot communication, and so forth.

Twenty-five questions offered a range of possible answers: excellent, good, fair, or poor. A response of excellent was given a score of 4, good a 3, fair a 2, and poor a 1. These responses were then totaled, so the total score was an indication of the satisfaction with the flight. Brunner Marketing Research, Inc., randomly selected and surveyed passengers from the four airlines.

Is there a difference in the mean satisfaction level among the four airlines?

Use the .01 significance level.

Eastern	TWA	Allegheny	Ozark
94	75	70	68
90	68	73	70
85	77	76	72
80	83	78	65
	88	80	74
		68	65
		65	

Step 1: State the null and alternate hypotheses.

H_0: $\mu_E = \mu_A = \mu_T = \mu_O$

H_1: The means are not all equal

Reject H_0 if $F > F_{\alpha,k-1,n-k}$

Step 2: State the level of significance.

The .01 significance level is stated in the problem.

Step 3: Find the appropriate test statistic.

Use the F statistic

Step 4: State the decision rule.

Reject H_0 if: $F > F_{\alpha,k-1,n-k}$

$F > F_{.01,4-1,22-4}$

$F > F_{.01,3,18}$

$F > 5.09$

Source of Variation	Sum of Squares	Degrees of Freedom	Mean Square	F
Treatments	890.68	3	296.89	8.99
Error	594.41	18	33.02	
Total	1,485.09	21		

The computed value of F is 8.99, which is greater than the critical value of 5.09, so the **null hypothesis is rejected**.

Conclusion: The mean scores are **not the same** for the four airlines; at this point we can **only conclude there is a difference in the treatment means**. We cannot determine which treatment groups differ or how many treatment groups differ.

Comparing Means of Two or More Populations – Example

Step 5: Compute the value of *F* and make a decision

ANOVA Table

Source of Variation	Sum of Squares	Degrees of Freedom	Mean Square	*F*
Treatments	SST	$k-1$	$SST/(k-1) = MST$	MST/MSE
Error	SSE	$n-k$	$SSE/(n-k) = MSE$	
Total	SS total	$n-1$		

$$\text{SS total} = \Sigma(X - \bar{X}_G)^2$$

where:
X is each sample observation.
$\bar{X}_G$ is the overall or grand mean.

$$SSE = \Sigma(X - \bar{X}_c)^2$$

where:
$\bar{X}_c$ is the sample mean for treatment c.

$$\bar{X}_G = \frac{1{,}664}{22} = 75.64$$

	Eastern	TWA	Allegheny	Ozark	Total
	94	75	70	68	
	90	68	73	70	
	85	77	76	72	
	80	83	78	65	
		88	80	74	
			68	65	
			65		
Column total	349	391	510	414	1,664
n	4	5	7	6	22
Mean	87.25	78.20	72.86	69.00	75.64

Confidence Interval for the Difference Between Two Means

When we reject the null hypothesis that the means are equal, we may want to know which treatment means differ. One of the simplest procedures is through the use of confidence intervals.

$$(\bar{X}_1 - \bar{X}_2) \pm t\sqrt{MSE\left(\frac{1}{n_1} + \frac{1}{n_2}\right)}$$

where
$\bar{X}_1$ is the mean of the first sample.
$\bar{X}_2$ is the mean of the second sample.
t is obtained from Appendix B.2. The degrees of freedom is equal to $n - k$.
MSE is the mean square error term obtained from the ANOVA table [SSE/($n - k$)].
n_1 is the number of observations in the first sample.
n_2 is the number of observations in the second sample.

EXAMPLE
From the previous example, develop a 95% confidence interval for the difference in the mean rating for Eastern and Ozark. Can we conclude that there is a difference between the two airlines' ratings?

$$(\bar{X}_E - \bar{X}_O) \pm t\sqrt{\text{MSE}\left(\frac{1}{n_E} + \frac{1}{n_O}\right)}$$

$$= (87.25 - 69.00) \pm 2.101\sqrt{33.0\left(\frac{1}{4} + \frac{1}{6}\right)}$$
$$= 18.25 \pm 7.79$$

The 95 percent confidence interval ranges from 10.46 up to 26.04. Both endpoints are positive; hence, we can conclude these treatment means differ significantly. That is, passengers on Eastern rated service significantly different from those on Ozark.