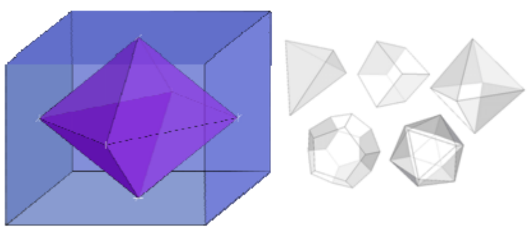
Lecture 11: Duality Theory and Sensitivity Analysis



Outline

- The essence of duality theory and its applications
- Duality theory for primal in our standard form
- Duality theory for primal in other forms
- Economic interpretation of the dual problem
- Relationships between the primal and dual problems
- The role of duality theory in sensitivity analysis

Concept of duality

- Every linear programming problem has associated with it another linear programming problem called the **dual**.
- The relationships between the dual problem and the original problem (called the **primal**) prove to be extremely useful in a variety of ways.
- Shadow prices actually are provided by the optimal solution for the dual problem.
- One key use of duality theory lies in the interpretation and implementation of *sensitivity analysis.*

The Essence of Duality Theory

Primal Problem	*Dual Problem*
Maximize $Z = \sum_{j=1}^{n} c_j x_j$,	Minimize $W = \sum_{i=1}^{m} b_i y_i$,
subject to	subject to
$\sum_{j=1}^{n} a_{ij} x_j \le b_i$, for $i = 1, 2, \ldots, m$	$\sum_{i=1}^{m} a_{ij} y_i \ge c_j$, for $j = 1, 2, \ldots, n$
and	and
$x_j \ge 0$, for $j = 1, 2, \ldots, n$.	$y_i \ge 0$, for $i = 1, 2, \ldots, m$.

The dual problem uses exactly the same *parameters as the primal problem, but in different* locations.

Matrix Notation

Primal Problem	*Dual Problem*
Maximize $Z = \mathbf{cx}$,	Minimize $W = \mathbf{yb}$,
subject to	subject to
$\mathbf{Ax} \le \mathbf{b}$	$\mathbf{yA} \ge \mathbf{c}$
and	and
$\mathbf{x} \ge \mathbf{0}$.	$\mathbf{y} \ge \mathbf{0}$.

c and ***y*** = $[y_1, y_2, \ldots, y_m]$ are row vectors but ***b*** and ***x*** are column vectors.

Primal and dual problems for Wyndor Glass Co. example

Primal Problem in Algebraic Form

Maximize $Z = 3x_1 + 5x_2$,

subject to

$$x_1 \le 4$$
$$2x_2 \le 12$$
$$3x_1 + 2x_2 \le 18$$

and $x_1 \ge 0, \quad x_2 \ge 0$.

Dual Problem in Algebraic Form

Minimize $W = 4y_1 + 12y_2 + 18y_3$,

subject to

$$y_1 + 3y_3 \ge 3$$
$$2y_2 + 2y_3 \ge 5$$

and

$$y_1 \ge 0, \quad y_2 \ge 0, \quad y_3 \ge 0.$$

Primal and dual problems for Wyndor Glass Co. example

Primal Problem in Matrix Form

Maximize $Z = [3, 5]\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$,

subject to

$$\begin{bmatrix} 1 & 0 \\ 0 & 2 \\ 3 & 2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \le \begin{bmatrix} 4 \\ 12 \\ 18 \end{bmatrix}$$

and

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \ge \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

Dual Problem in Matrix Form

Minimize $W = [y_1, y_2, y_3]\begin{bmatrix} 4 \\ 12 \\ 18 \end{bmatrix}$

subject to

$$[y_1, y_2, y_3]\begin{bmatrix} 1 & 0 \\ 0 & 2 \\ 3 & 2 \end{bmatrix} \ge [3, 5]$$

and

$$[y_1, y_2, y_3] \ge [0, 0, 0].$$

Primal-dual table

			Primal Problem					
			Coefficient of:				Right Side	
			x_1	x_2	$\cdots$	x_n		
Dual Problem	Coefficient of:	y_1	a_{11}	a_{12}	$\cdots$	a_{1n}	$\le b_1$	Coefficients for Objective Function (Minimize)
		y_2	a_{21}	a_{22}	$\cdots$	a_{2n}	$\le b_2$	
		$\vdots$					$\vdots$	
		y_m	a_{m1}	a_{m2}	$\cdots$	a_{mn}	$\le b_m$	
		Right Side	VI	VI	$\cdots$	VI		
			c_1	c_2	$\cdots$	c_n		
			Coefficients for Objective Function (Maximize)					

Primal-dual table for Wyndor Problem

	x_1	x_2	
y_1	1	0	≤ 4
y_2	0	2	≤ 12
y_3	3	2	≤ 18
	$\geq$ 3	$\geq$ 5	

Correspondence between entities in primal and dual problems

One Problem	Other Problem
Constraint i ⟷	Variable i
Objective function ⟷	Right sides

Fundamental Insight

Primal Problem	*Dual Problem*
Maximize $Z = \mathbf{cx}$,	Minimize $W = \mathbf{yb}$,
subject to	subject to
$\mathbf{Ax} \leq \mathbf{b}$	$\mathbf{yA} \geq \mathbf{c}$
and	and
$\mathbf{x} \geq \mathbf{0}$.	$\mathbf{y} \geq \mathbf{0}$.

$$W = \mathbf{yb} = \sum_{i=1}^{m} b_i y_i,$$

$$\mathbf{z} = \mathbf{yA}, \quad \text{so} \quad z_j = \sum_{i=1}^{m} a_{ij} y_i, \quad \text{for } j = 1, 2, \ldots, n.$$

(z_1 - c_1) and (z_2 - c_2) can be interpreted as being the surplus variables for these **functional constraints.**

Notation for entries in row 0 of a simplex tableau

Iteration	Basic Variable	Eq.	Coefficient of: Z	x_1	x_2	$\cdots$	x_n	x_{n+1}	x_{n+2}	$\cdots$	x_{n+m}	Right Side
Any	Z	(0)	1	$z_1 - c_1$	$z_2 - c_2$	$\cdots$	$z_n - c_n$	y_1	y_2	$\cdots$	y_m	W

z ***denotes the vector that the simplex method added to the*** **vector of** ***initial coefficients, −c, in the process of reaching the current tableau***

Condition for Optimality

Iteration	Basic Variable	Eq.	Coefficient of: Z	x_1	x_2	$\cdots$	x_n	x_{n+1}	x_{n+2}	$\cdots$	x_{n+m}	Right Side
Any	Z	(0)	1	$z_1 - c_1$	$z_2 - c_2$	$\cdots$	$z_n - c_n$	y_1	y_2	$\cdots$	y_m	W

$$\boxed{\begin{aligned} z_j - c_j &\geq 0 \qquad \text{for } j = 1, 2, \ldots, n, \\ y_i &\geq 0 \qquad \text{for } i = 1, 2, \ldots, m. \end{aligned}} \longrightarrow$$

Constraints of dual problem

$$W = \sum_{i=1}^{m} b_i y_i,$$

subject to

$$\sum_{i=1}^{m} a_{ij} y_i \geq c_j, \qquad \text{for } j = 1, 2, \ldots, n$$

and

$$y_i \geq 0, \qquad \text{for } i = 1, 2, \ldots, m.$$

Maximize or minimize?

$$W = \sum_{i=1}^{m} b_i y_i,$$

subject to

$$\sum_{i=1}^{m} a_{ij} y_i \geq c_j, \qquad \text{for } j = 1, 2, \ldots, n$$

and

$$y_i \geq 0, \qquad \text{for } i = 1, 2, \ldots, m.$$

- It is *only the optimal solution for the primal problem that corresponds to a feasible* solution for this new problem.
- As a consequence, the optimal value of *Z in the primal* problem is the *minimum feasible value of W in the new problem.*

Wyndor Glass Co. example

$$\text{Minimize} \qquad W = 4y_1 + 12y_2 + 18y_3,$$

subject to

$$\begin{aligned} y_1 \quad\;\; + 3y_3 - (z_1 - c_1) &= 3 \\ 2y_2 + 2y_3 - (z_2 - c_2) &= 5 \end{aligned}$$

and

$$y_1 \geq 0, \qquad y_2 \geq 0, \qquad y_3 \geq 0.$$

Row 0 and corresponding dual solution

Iteration	Primal Problem: Row 0	Dual Problem: y_1	y_2	y_3	$z_1 - c_1$	$z_2 - c_2$	W
0	$[-3, \; -5 \mid 0, \; 0, \; 0 \mid 0]$						
1	$[-3, \; 0 \mid 0, \; \frac{5}{2}, \; 0 \mid 30]$						
2	$[\;0, \; 0 \mid 0, \; \frac{3}{2}, \; 1 \mid 36]$						

Summary of Primal-Dual Relationships

- **Weak duality property**: If ***x*** is a feasible solution for the primal problem and ***y*** is a feasible solution for the dual problem, then $\boldsymbol{cx} \leq \boldsymbol{yb}$.
- **Strong duality property**: If ***x**** is an optimal solution for the primal problem and ***y**** is an optimal solution for the dual problem, then $\boldsymbol{cx}^* = \boldsymbol{y}^*\boldsymbol{b}$.
- **Complementary solutions property**: At each iteration, the simplex method simultaneously identifies a CPF solution ***x*** for the primal problem and a complementary solution ***y*** for the dual problem, where $\boldsymbol{cx} = \boldsymbol{yb}$.

Summary of Primal-Dual Relationships

- **Complementary optimal solutions property**: At the final iteration, the simplex method simultaneously identifies an optimal solution ***x**** for the primal problem and a complementary optimal solution ***y**** for the dual problem, where $\boldsymbol{cx}^* = \boldsymbol{y}^*\boldsymbol{b}$.
- **Symmetry property**: For *any primal problem and its dual problem, all relationships* between them must be *symmetric because the dual of this dual problem* is this primal problem.

Duality theorem

1. If one problem has *feasible solutions and a bounded objective function (and* so has an optimal solution), then so does the other problem, so both the weak and strong duality properties are applicable.
2. If one problem has *feasible solutions and an unbounded objective function* (and so *no optimal solution), then the other problem has no feasible solutions.*
3. If one problem has *no feasible solutions, then the other problem has either no feasible solutions or an unbounded objective function.*

Economic Interpretation of Duality

$$W = b_1y_1 + b_2y_2 + \cdots + b_my_m$$

- Each b_iy_i can be interpreted as the current contribution to profit by having b_i units of resource i available for the primal problem.

Quantity	Interpretation
x_j	Level of activity j $(j = 1, 2, \ldots, n)$
c_j	Unit profit from activity j
Z	Total profit from all activities
b_i	Amount of resource i available $(i = 1, 2, \ldots, m)$
a_{ij}	Amount of resource i consumed by each unit of activity j

Primal-dual Relationships

$$z_j - c_j = \sum_{i=1}^{m} a_{ij} y_i - c_j, \qquad \text{for } j = 1, 2, \ldots, n.$$

- *m* and *n* reverse their previous roles in dual problem.

Complementary Basic Solutions

	Primal Variable	Associated Dual Variable
Any problem	(Decision variable) x_j (Slack variable) x_{n+i}	$z_j - c_j$ (surplus variable) $j = 1, 2, \ldots, n$ y_i (decision variable) $i = 1, 2, \ldots, m$
Wyndor problem	Decision variables: x_1 x_2 Slack variables: x_3 x_4 x_5	$z_1 - c_1$ (surplus variables) $z_2 - c_2$ y_1 (decision variables) y_2 y_3

Complementary Basic Solutions

- **Complementary basic solutions property:** Each *basic solution in the primal problem has a complementary basic solution in the dual problem, where their* respective objective function values (*Z and W) are equal. Given row 0 of the simplex* tableau for the primal basic solution, the complementary dual basic solution (***y***, ***z*** - ***c***) is found.
- **Complementary slackness property:** Given the association between variables in previous table, the variables in the primal basic solution and the complementary dual basic solution satisfy the complementary slackness relationship shown in following table. Furthermore, this relationship is a symmetric one, so that these two basic solutions are complementary to each other.

Complementary slackness relationship for complementary basic solutions

Primal Variable	Associated Dual Variable	
Basic	Nonbasic	(*m* variables)
Nonbasic	Basic	(*n* variables)

Complementary basic solutions for Wyndor Glass Co. example

	Primal Problem			Dual Problem	
No.	Basic Solution	Feasible?	$Z = W$	Feasible?	Basic Solution
1	(0, 0, 4, 12, 18)	Yes	0	No	(0, 0, 0, −3, −5)
2	(4, 0, 0, 12, 6)	Yes	12	No	(3, 0, 0, 0, −5)
3	(6, 0, −2, 12, 0)	No	18	No	(0, 0, 1, 0, −3)
4	(4, 3, 0, 6, 0)	Yes	27	No	$\left(-\frac{9}{2}, 0, \frac{5}{2}, 0, 0\right)$
5	(0, 6, 4, 0, 6)	Yes	30	No	$\left(0, \frac{5}{2}, 0, -3, 0\right)$
6	(2, 6, 2, 0, 0)	Yes	36	Yes	$\left(0, \frac{3}{2}, 1, 0, 0\right)$
7	(4, 6, 0, 0, −6)	No	42	Yes	$\left(3, \frac{5}{2}, 0, 0, 0\right)$
8	(0, 9, 4, −6, 0)	No	45	Yes	$\left(0, 0, \frac{5}{2}, \frac{9}{2}, 0\right)$

Relationships between Complementary Basic Solutions

- **Complementary optimal basic solutions property:** Each *optimal basic solution* in the *primal problem has a complementary optimal basic solution in the dual* problem, where their respective objective function values (*Z and W) are equal.* Given row 0 of the simplex tableau for the optimal primal solution, the complementary optimal dual solution ($\mathbf{y}^*$, $\mathbf{z}^* - \mathbf{c}$) is found.

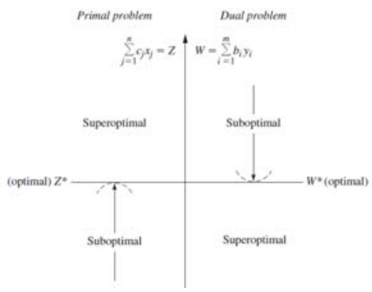
Classification of basic solutions

		Satisfies Condition for Optimality?	
		Yes	No
Feasible?	Yes	Optimal	Suboptimal
	No	Superoptimal	Neither feasible nor superoptimal

Relationships between complementary basic solutions

Primal Basic Solution	Complementary Dual Basic Solution	Both Basic Solutions	
		Primal Feasible?	Dual Feasible?
Suboptimal	Superoptimal	Yes	No
Optimal	Optimal	Yes	Yes
Superoptimal	Suboptimal	No	Yes
Neither feasible nor superoptimal	Neither feasible nor superoptimal	No	No

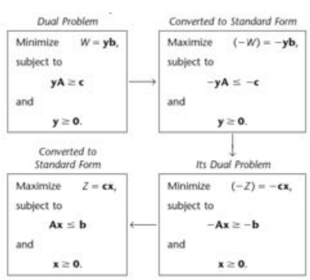
Range of possible values of complementary basic solutions



Adapting to Other Primal Forms

Nonstandard Form	Equivalent Standard Form
Minimize Z	Maximize $(-Z)$
$\sum_{j=1}^{n} a_{ij}x_j \geq b_i$	$-\sum_{j=1}^{n} a_{ij}x_j \leq -b_i$
$\sum_{j=1}^{n} a_{ij}x_j = b_i$	$\sum_{j=1}^{n} a_{ij}x_j \leq b_i$ and $-\sum_{j=1}^{n} a_{ij}x_j \leq -b_i$
x_j unconstrained in sign	$x_j^+ - x_j^-$, $x_j^+ \geq 0$, $x_j^- \geq 0$

Constructing the dual of the dual problem



SOB Method for Determining Form of Constraints in Dual

Label	Primal Problem (or Dual Problem)	Dual Problem (or Primal Problem)
	Maximize Z (or W)	Minimize W (or Z)
	Constraint i:	Variable y_i (or x_i):
Sensible	$\leq$ form	$y_i \geq 0$
Odd	$=$ form	Unconstrained
Bizarre	$\geq$ form	$y_i' \leq 0$
	Variable x_j (or y_j):	Constraint j:
Sensible	$x_j \geq 0$	$\geq$ form
Odd	Unconstrained	$=$ form
Bizarre	$x_j' \leq 0$	$\leq$ form

One primal-dual form for radiation therapy example

Primal Problem

Maximize $-Z = -0.4x_1 - 0.5x_2,$

subject to

(S) $0.3x_1 + 0.1x_2 \leq 2.7$

(O) $0.5x_1 + 0.5x_2 = 6$

(B) $0.6x_1 + 0.4x_2 \geq 6$

and

(S) $x_1 \geq 0$

(S) $x_2 \geq 0$

Dual Problem

Another primal-dual form for radiation therapy example

Primal Problem

Minimize $Z = 0.4x_1 + 0.5x_2,$

subject to

(B) $0.3x_1 + 0.1x_2 \leq 2.7$

(O) $0.5x_1 + 0.5x_2 = 6$

(S) $0.6x_1 + 0.4x_2 \geq 6$

and

(S) $x_1 \geq 0$

(S) $x_2 \geq 0$

Dual Problem

Role of Duality Theory in Sensitivity Analysis

- Sensitivity analysis involves investigating the effect on the optimal solution of making changes in the values of the model parameters a_{ij}, b_i, *and* c_j.
- Changing parameter values in the primal problem also changes the corresponding values in the dual problem.
- You can deal with either primal or dual.

Changes in Coefficients of a Nonbasic Variable

- Suppose that the changes made in the original model occur in the coefficients of a variable that was nonbasic in the original optimal solution.
- What is the effect of these changes on this solution?
  - Is it still feasible? Is it still optimal?

Introduction of a New Variable

- Would adding any new variable (activity) to the model change the original optimal solution?

Maximize $\quad Z = 3x_1 + 5x_2 + 4x_{new}$,

subject to

$$\begin{aligned} x_1 \quad\quad + 2x_{new} &\le 4 \\ 2x_2 + 3x_{new} &\le 12 \\ 3x_1 + 2x_2 + x_{new} &\le 18 \end{aligned}$$

and

$$x_1 \ge 0, \quad x_2 \ge 0, \quad x_{new} \ge 0.$$

The Essence of Sensitivity Analysis

- Revised model, $\mathbf{b} \to \bar{\mathbf{b}}, \quad \mathbf{c} \to \bar{\mathbf{c}}, \quad \mathbf{A} \to \bar{\mathbf{A}}$
- The first step is to revise the final simplex tableau to reflect these changes.

	Eq.	Z	Original Variables	Slack Variables	Right Side
		Coefficient of:			
New initial tableau	(0)	1	$-\bar{\mathbf{c}}$	$\mathbf{0}$	0
	(1, 2, . . . , m)	$\mathbf{0}$	$\bar{\mathbf{A}}$	$\mathbf{I}$	$\bar{\mathbf{b}}$
Revised final tableau	(0)	1	$\mathbf{z}^* - \bar{\mathbf{c}} = \mathbf{y}^*\bar{\mathbf{A}} - \bar{\mathbf{c}}$	$\mathbf{y}^*$	$Z^* = \mathbf{y}^*\bar{\mathbf{b}}$
	(1, 2, . . . , m)	$\mathbf{0}$	$\mathbf{A}^* = \mathbf{S}^*\bar{\mathbf{A}}$	$\mathbf{S}^*$	$\mathbf{b}^* = \mathbf{S}^*\bar{\mathbf{b}}$

Variation 1 of the Wyndor Model

Original Model

Maximize $\quad Z = [3, 5]\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$,

subject to

$$\begin{bmatrix} 1 & 0 \\ 0 & 2 \\ 3 & 2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \le \begin{bmatrix} 4 \\ 12 \\ 18 \end{bmatrix}$$

and

$$\mathbf{x} \ge \mathbf{0}.$$

Revised Model

Maximize $\quad Z = [4, 5]\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$,

subject to

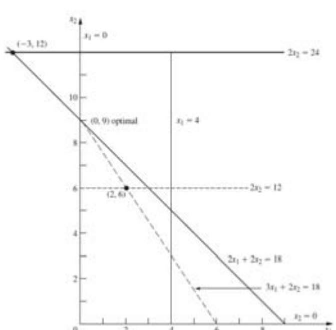
$$\begin{bmatrix} 1 & 0 \\ 0 & 2 \\ 2 & 2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \le \begin{bmatrix} 4 \\ 24 \\ 18 \end{bmatrix}$$

and

$$\mathbf{x} \ge \mathbf{0}.$$

$$c_1 = 3 \to 4, \; a_{31} = 3 \to 2, \; b_2 = 12 \to 24.$$

Graphical Solution



Fundamental Insight

$$\bar{\mathbf{c}} = [4, 5], \qquad \bar{\mathbf{A}} = \begin{bmatrix} 1 & 0 \\ 0 & 2 \\ 2 & 2 \end{bmatrix}, \qquad \bar{\mathbf{b}} = \begin{bmatrix} 4 \\ 24 \\ 18 \end{bmatrix}.$$

	Basic Variable	Eq.	Coefficient of: Z	x_1	x_2	x_3	x_4	x_5	Right Side
New initial tableau	Z	(0)	1	−4	−5	0	0	0	0
	x_3	(1)	0	1	0	1	0	0	4
	x_4	(2)	0	0	2	0	1	0	24
	x_5	(3)	0	2	2	0	0	1	18
Final tableau for original model	Z	(0)	1	0	0	0	$\frac{3}{2}$	1	36
	x_3	(1)	0	0	0	1	$\frac{1}{3}$	$-\frac{1}{3}$	2
	x_2	(2)	0	0	1	0	$\frac{1}{2}$	0	6
	x_1	(3)	0	1	0	0	$-\frac{1}{3}$	$\frac{1}{3}$	2

Fundamental Insight

$$\mathbf{y}^* = [0, \tfrac{3}{2}, 1], \qquad \mathbf{S}^* = \begin{bmatrix} 1 & \frac{1}{3} & -\frac{1}{3} \\ 0 & \frac{1}{2} & 0 \\ 0 & -\frac{1}{3} & \frac{1}{3} \end{bmatrix}.$$

$$\mathbf{z}^* - \bar{\mathbf{c}} = [0, \tfrac{3}{2}, 1]\begin{bmatrix} 1 & 0 \\ 0 & 2 \\ 2 & 2 \end{bmatrix} - [4, 5] = [-2, 0], \qquad Z^* = [0, \tfrac{3}{2}, 1]\begin{bmatrix} 4 \\ 24 \\ 18 \end{bmatrix} = 54,$$

$$\mathbf{A}^* = \begin{bmatrix} 1 & \frac{1}{3} & -\frac{1}{3} \\ 0 & \frac{1}{2} & 0 \\ 0 & -\frac{1}{3} & \frac{1}{3} \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 2 \\ 2 & 2 \end{bmatrix} = \begin{bmatrix} \frac{1}{3} & 0 \\ 0 & 1 \\ \frac{2}{3} & 0 \end{bmatrix},$$

$$\mathbf{b}^* = \begin{bmatrix} 1 & \frac{1}{3} & -\frac{1}{3} \\ 0 & \frac{1}{2} & 0 \\ 0 & -\frac{1}{3} & \frac{1}{3} \end{bmatrix}\begin{bmatrix} 4 \\ 24 \\ 18 \end{bmatrix} = \begin{bmatrix} 6 \\ 12 \\ -2 \end{bmatrix}.$$

Revised Final Tableau

Final tableau for original model	Z	(0)	1	0	0	0	$\frac{3}{2}$	1	36
	x_3	(1)	0	0	0	1	$\frac{1}{3}$	$-\frac{1}{3}$	2
	x_2	(2)	0	0	1	0	$\frac{1}{2}$	0	6
	x_1	(3)	0	1	0	0	$-\frac{1}{3}$	$\frac{1}{3}$	2
Revised final tableau	Z	(0)	1	−2	0	0	$\frac{3}{2}$	1	54
	x_3	(1)	0	$\frac{1}{3}$	0	1	$\frac{1}{3}$	$-\frac{1}{3}$	6
	x_2	(2)	0	0	1	0	$\frac{1}{2}$	0	12
	x_1	(3)	0	$\frac{2}{3}$	0	0	$-\frac{1}{3}$	$\frac{1}{3}$	−2

Calculating only incremental changes in final tableau

- Only incremental changes, $\Delta c_1 = 1$, $\Delta a_{31} = -1$, and $\Delta b_2 = 12$
- A zero or dash appearing in each spot indicates no calculation is needed.

$$\Delta(\mathbf{z}^* - \mathbf{c}) = \mathbf{y}^* \, \Delta\mathbf{A} - \Delta\mathbf{c} = [0, \tfrac{3}{2}, 1]\begin{bmatrix} 0 & — \\ 0 & — \\ -1 & — \end{bmatrix} - [1, —] = [-2, —].$$

$$\Delta Z^* = \mathbf{y}^* \, \Delta\mathbf{b} = [0, \tfrac{3}{2}, 1]\begin{bmatrix} 0 \\ 12 \\ 0 \end{bmatrix} = 18.$$

$$\Delta\mathbf{A}^* = \mathbf{S}^* \, \Delta\mathbf{A} = \begin{bmatrix} 1 & \frac{1}{3} & -\frac{1}{3} \\ 0 & \frac{1}{2} & 0 \\ 0 & -\frac{1}{3} & \frac{1}{3} \end{bmatrix}\begin{bmatrix} 0 & — \\ 0 & — \\ -1 & — \end{bmatrix} = \begin{bmatrix} \frac{1}{3} & — \\ 0 & — \\ -\frac{1}{3} & — \end{bmatrix}.$$

$$\Delta\mathbf{b}^* = \mathbf{S}^* \, \Delta\mathbf{b} = \begin{bmatrix} 1 & \frac{1}{3} & -\frac{1}{3} \\ 0 & \frac{1}{2} & 0 \\ 0 & -\frac{1}{3} & \frac{1}{3} \end{bmatrix}\begin{bmatrix} 0 \\ 12 \\ 0 \end{bmatrix} = \begin{bmatrix} 4 \\ 6 \\ -4 \end{bmatrix}.$$

Converting revised final simplex tableau to proper form

	Basic Variable	Eq.	Coefficient of: Z	x_1	x_2	x_3	x_4	x_5	Right Side
Revised final tableau	Z	(0)	1	-2	0	0	$\frac{3}{2}$	1	54
	x_3	(1)	0	$\frac{1}{3}$	0	1	$\frac{1}{3}$	$-\frac{1}{3}$	6
	x_2	(2)	0	0	1	0	$\frac{1}{2}$	0	12
	x_1	(3)	0	$\frac{2}{3}$	0	0	$-\frac{1}{3}$	$\frac{1}{3}$	-2
Converted to proper form	Z	(0)	1	0	0	0	$\frac{1}{2}$	2	48
	x_3	(1)	0	0	0	1	$\frac{1}{2}$	$-\frac{1}{2}$	7
	x_2	(2)	0	0	1	0	$\frac{1}{2}$	0	12
	x_1	(3)	0	1	0	0	$-\frac{1}{2}$	$\frac{1}{2}$	-3

Summary of Procedure for Sensitivity Analysis

1. Revision of model
2. Revision of final tableau
3. Conversion to proper form from Gaussian elimination
4. Feasibility test
5. Optimality test
6. Reoptimization

Applying Sensitivity Analysis

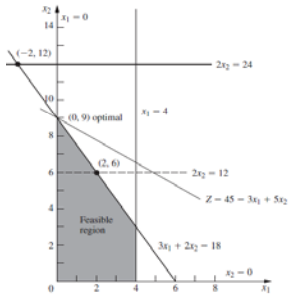
- Case 1—Changes in b_i
- Case 2a—Changes in the Coefficients of a Nonbasic Variable
- Case 2b—Introduction of a New Variable
- Case 3—Changes in the Coefficients of a Basic Variable
- Case 4—Introduction of a New Constraint

Case 1—Changes in b_i

- Both the conversion to proper form from Gaussian elimination and the optimality test steps of the general procedure can be skipped.

Right side of final row 0: $Z^* = \mathbf{y}^*\bar{\mathbf{b}}$,
Right side of final rows 1, 2, . . . , m: $\mathbf{b}^* = \mathbf{S}^*\bar{\mathbf{b}}$.

Example (Variation 2 of the Wyndor Model)



$$\mathbf{b} = \begin{bmatrix} 4 \\ 12 \\ 18 \end{bmatrix} \longrightarrow \bar{\mathbf{b}} = \begin{bmatrix} 4 \\ 24 \\ 18 \end{bmatrix}.$$

Analysis of Variation 2

$$Z^* = \mathbf{y}^*\bar{\mathbf{b}} = [0, \tfrac{3}{2}, 1]\begin{bmatrix} 4 \\ 24 \\ 18 \end{bmatrix} = 54,$$

$$\mathbf{b}^* = \mathbf{S}^*\bar{\mathbf{b}} = \begin{bmatrix} 1 & \frac{1}{3} & -\frac{1}{3} \\ 0 & \frac{1}{2} & 0 \\ 0 & -\frac{1}{3} & \frac{1}{3} \end{bmatrix}\begin{bmatrix} 4 \\ 24 \\ 18 \end{bmatrix} = \begin{bmatrix} 6 \\ 12 \\ -2 \end{bmatrix}, \quad \text{so} \begin{bmatrix} x_3 \\ x_2 \\ x_1 \end{bmatrix} = \begin{bmatrix} 6 \\ 12 \\ -2 \end{bmatrix}.$$

$$\Delta Z^* = \mathbf{y}^*\Delta\mathbf{b} = \mathbf{y}^*\begin{bmatrix} \Delta b_1 \\ \Delta b_2 \\ \Delta b_3 \end{bmatrix} = \mathbf{y}^*\begin{bmatrix} 0 \\ 12 \\ 0 \end{bmatrix}, \quad \Delta\mathbf{b}^* = \mathbf{S}^*\,\Delta\mathbf{b} = \mathbf{S}^*\begin{bmatrix} \Delta b_1 \\ \Delta b_2 \\ \Delta b_3 \end{bmatrix} = \mathbf{S}^*\begin{bmatrix} 0 \\ 12 \\ 0 \end{bmatrix}$$

$$\Delta Z^* = \frac{3}{2}(12) = 18, \qquad \text{so } Z^* = 36 + 18 = 54,$$

$$\Delta b_1^* = \frac{1}{3}(12) = 4, \qquad \text{so } b_1^* = 2 + 4 = 6,$$

$$\Delta b_2^* = \frac{1}{2}(12) = 6, \qquad \text{so } b_2^* = 6 + 6 = 12,$$

$$\Delta b_3^* = -\frac{1}{3}(12) = -4, \qquad \text{so } b_3^* = 2 - 4 = -2.$$

Final Simplex Tableau after Reoptimization

Model Parameters

$c_1 = 3,$	$c_2 = 5$	$(n = 2)$
$a_{11} = 1,$	$a_{12} = 0,$	$b_1 = 4$
$a_{21} = 0,$	$a_{22} = 2,$	$b_2 = 24$
$a_{31} = 3,$	$a_{32} = 2,$	$b_3 = 18$

Basic Variable	Eq.	Coefficient of:						Right Side
		Z	x_1	x_2	x_3	x_4	x_5	
Z	(0)	1	$\frac{9}{2}$	0	0	0	$\frac{5}{2}$	45
x_3	(1)	0	1	0	1	0	0	4
x_2	(2)	0	$\frac{3}{2}$	1	0	0	$\frac{1}{2}$	9
x_4	(3)	0	−3	0	0	1	−1	6

Allowable Range to Stay Feasible

$$b_1^* = 2 + \frac{1}{3}\Delta b_2,$$

$$b_2^* = 6 + \frac{1}{2}\Delta b_2,$$

$$b_3^* = 2 - \frac{1}{3}\Delta b_2,$$

- The solution remains feasible, and so optimal, as long as all three quantities remain nonnegative.

$$2 + \frac{1}{3}\Delta b_2 \geq 0 \Rightarrow \frac{1}{3}\Delta b_2 \geq -2 \Rightarrow \Delta b_2 \geq -6,$$

$$6 + \frac{1}{2}\Delta b_2 \geq 0 \Rightarrow \frac{1}{2}\Delta b_2 \geq -6 \Rightarrow \Delta b_2 \geq -12,$$

$$2 - \frac{1}{3}\Delta b_2 \geq 0 \Rightarrow 2 \geq \frac{1}{3}\Delta b_2 \Rightarrow \Delta b_2 \leq 6.$$

Allowable Range to Stay Feasible

Therefore, since $b_2 = 12 + \Delta b_2$, the solution remains feasible only if

$$-6 \le \Delta b_2 \le 6, \quad \text{that is,} \quad 6 \le b_2 \le 18.$$

- This range of values for b_2 is referred to as its *allowable range to stay feasible*.
- For any b_i, its allowable range to stay feasible is the range of values over which the current optimal BF solution (with adjusted values for the basic variables) remains feasible.
- Thus, the shadow price for b_i remains valid as long as b_i remains within this allowable range.

Analyzing Simultaneous Changes in Right-Hand Sides

- For each b_i, the allowable range to stay feasible gives this range if none of the other b_i are changing at the same time. What do these allowable ranges become when some of the b_i are changing simultaneously?

The 100 Percent Rule for Simultaneous Changes in Right-Hand Sides

- The shadow prices remain valid for predicting the effect of simultaneously changing the right-hand sides of some of the functional constraints as long as the changes are not too large.
- To check whether the changes are small enough, calculate for each change the percentage of the allowable change (increase or decrease) for that right-hand side to remain within its allowable range to stay feasible.
- If the sum of the percentage changes does not exceed 100 percent, the shadow prices definitely will still be valid. (If the sum does exceed 100 percent, then we cannot be sure.)

Example (Variation 3 of the Wyndor Model)

$$\mathbf{b} = \begin{bmatrix} 4 \\ 12 \\ 18 \end{bmatrix} \rightarrow \bar{\mathbf{b}} = \begin{bmatrix} 4 \\ 15 \\ 15 \end{bmatrix}.$$

Constraint	Shadow Price	Current RHS	Allowable Increase	Allowable Decrease
Plant 1	0	4	∞	2
Plant 2	1.5	12	6	6
Plant 3	1	18	6	6

b_2: $12 \rightarrow 15$. Percentage of allowable increase $= 100\left(\frac{15-12}{6}\right) = 50\%$

b_3: $18 \rightarrow 15$. Percentage of allowable decrease $= 100\left(\frac{18-15}{6}\right) = 50\%$

Sum $= 100\%$

Assignment

- Problems 6.1-1, 6.1-3, 6.1-4, 6.1-5, 6.1-8, 6.1-13, 6.3-5, 6.3-6, 6.4-1, 6.4-2, 6.6-1, 6.6-2, 6.6-3, 6.7-1.