

BANGLADESH ARMY INTERNATIONAL UNIVERSITY OF SCIENCE AND TECHNOLOGY (BAIUST), CUMILLA

Mid Term Examination, Fall 2023

Department of Computer Science and Engineering (CSE)

Level-1, Term-1

Course Code: CSE 113

Course Title: Discrete Mathematics

Credit Hour: 03

Full Marks: 90

Time: 1 hr. 30 mins

ANSWER SHEET

Examiner's Note: Answer any three (03) of the following four (04) questions including Q.No.-1.

Student's Note: All four questions are answered below for reference and study purposes.

SOLUTION TO QUESTION 1

a: Show that $\neg(p \vee (\neg p \wedge q))$ and $\neg p \wedge \neg q$ are logically equivalent by developing a series of logical equivalences.

Solution: We start with the left-hand side (LHS) and apply logical equivalences to transform it into the right-hand side (RHS).

$$\neg(p \vee (\neg p \wedge q))$$

$$1. \equiv \neg((p \vee \neg p) \wedge (p \vee q)) \quad (\text{Distributive Law})$$

$$2. \equiv \neg(T \wedge (p \vee q)) \quad (\text{Negation Law: } p \vee \neg p \equiv T)$$

$$3. \equiv \neg(p \vee q) \quad (\text{Identity Law: } T \wedge A \equiv A)$$

$$4. \equiv \neg p \wedge \neg q \quad (\text{De Morgan's Law})$$

Since we have transformed the LHS into the RHS, $\neg p \wedge \neg q$, the two propositions are logically equivalent. **(Proven)**

b: What are the contrapositive, the converse, and the inverse of the conditional statement "The home team wins whenever it is raining"?

Solution: First, we identify the propositions p and q and write the conditional statement in logical form. The statement "A whenever B" is logically equivalent to "If B, then A," or $B \rightarrow A$.

- p : It is raining.
- q : The home team wins.
- **Conditional Statement:** $p \rightarrow q$ (If it is raining, then the home team wins.)

Now, we define and state the required forms:

1. **Contrapositive** ($\neg q \rightarrow \neg p$):

- **Logical Form:** If the home team does not win, then it is not raining.

2. **Converse** ($q \rightarrow p$):

- **Logical Form:** If the home team wins, then it is raining.

3. **Inverse** ($\neg p \rightarrow \neg q$):

- **Logical Form:** If it is not raining, then the home team does not win.
-

c: Write the following propositions using p and q and logical connectives.

Solution: Given the propositions:

- p : You have the flu.
- q : You miss the final examination.

1. **If you have the flu, you will miss the final examination.**

$$p \rightarrow q$$

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2. You haven't any flu and you don't miss the final examination.

$$\neg p \wedge \neg q$$

3. You have the flu and you will miss the final examination.

$$p \wedge q$$

d: Construct a truth table for each of the following compound propositions.

i. $(p \leftrightarrow q) \vee (\neg p \leftrightarrow \neg r)$

Solution: We need 8 rows since there are three variables (p, q, r) .

p	q	r	$\neg p$	$\neg r$	$p \leftrightarrow q$	$\neg p \leftrightarrow \neg r$	$(p \leftrightarrow q) \vee (\neg p \leftrightarrow \neg r)$
T	T	T	F	F	T	T	T
T	T	F	F	T	T	F	T
T	F	T	F	F	F	T	T
T	F	F	F	T	F	F	F
F	T	T	T	F	F	F	F
F	T	F	T	T	F	T	T
F	F	T	T	F	T	F	T
F	F	F	T	T	T	T	T

ii. $p \rightarrow (\neg q \wedge r)$

Solution:

p	q	r	$\neg q$	$\neg q \wedge r$	$p \rightarrow (\neg q \wedge r)$
T	T	T	F	F	F
T	T	F	F	F	F
T	F	T	T	T	T
T	F	F	T	F	F
F	T	T	F	F	T
F	T	F	F	F	T
F	F	T	T	T	T
F	F	F	T	F	T

iii. $(p \leftrightarrow q) \oplus (p \leftrightarrow \neg q)$

Solution: Note: The \oplus symbol represents the *Exclusive OR* (XOR) operation.

p	q	$\neg q$	$p \leftrightarrow q$	$p \leftrightarrow \neg q$	$(p \leftrightarrow q) \oplus (p \leftrightarrow \neg q)$
T	T	F	T	F	T
T	F	T	F	T	T
F	T	F	F	T	T
F	F	T	T	F	T

SOLUTION TO QUESTION 2

a: Show that $A \times B \neq B \times A$, when A and B are nonempty unless $A = B$.

Solution: Let A and B be two non-empty sets.

- The Cartesian product $A \times B$ is the set of all ordered pairs (a, b) where $a \in A$ and $b \in B$.
- The Cartesian product $B \times A$ is the set of all ordered pairs (b, a) where $b \in B$ and $a \in A$.

For $A \times B = B \times A$, every element (a, b) in $A \times B$ must also be in $B \times A$. This requires that for any $a \in A$ and $b \in B$, the ordered pair (a, b) must be equal to an ordered pair in $B \times A$, which means $(a, b) = (b', a')$ where $b' \in B$ and $a' \in A$.

Consider a specific pair $(a, b) \in A \times B$. If $A \times B = B \times A$, then (a, b) must be an element of $B \times A$. The elements of $B \times A$ are of the form (x, y) where $x \in B$ and $y \in A$. For two ordered pairs to be equal, their corresponding elements must be equal:

$$(a, b) \in B \times A \implies a \in B \text{ and } b \in A$$

If $A \times B = B \times A$, this must hold true for **all** $a \in A$ and **all** $b \in B$.

If $a \in B$ for all $a \in A$, then $A \subseteq B$. If $b \in A$ for all $b \in B$, then $B \subseteq A$.

By the definition of set equality, $A = B$ if and only if $A \subseteq B$ and $B \subseteq A$. Therefore, $A \times B = B \times A$ if and only if $A = B$.

If $A \neq B$, then there exists at least one element $a \in A$ such that $a \notin B$, or there exists $b \in B$ such that $b \notin A$. **Case Example:** Let $A = \{1\}$ and $B = \{2\}$.

- $A \times B = \{(1, 2)\}$
- $B \times A = \{(2, 1)\}$

Since ordered pairs are equal if and only if their corresponding components are equal, $(1, 2) \neq (2, 1)$, and thus $A \times B \neq B \times A$. **(Proven)**

b: Let $f(x) = ax + b$ and $g(x) = cx + d$. Determine for which constants a, b, c , and d it is true that $f \circ g = g \circ f$.

Solution: We first compute the composite functions $f \circ g$ and $g \circ f$.

1. Compute $f \circ g(x)$

$$\begin{aligned} f \circ g(x) &= f(g(x)) = f(cx + d) \\ f(cx + d) &= a(cx + d) + b = acx + ad + b \end{aligned}$$

2. Compute $g \circ f(x)$

$$\begin{aligned}g \circ f(x) &= g(f(x)) = g(ax + b) \\g(ax + b) &= c(ax + b) + d = cax + cb + d\end{aligned}$$

3. Set $f \circ g(x) = g \circ f(x)$

We set the results equal:

$$acx + ad + b = cax + cb + d$$

Since $acx = cax$, the terms involving x are always equal. For the two functions to be equal for all x , the constant terms must be equal:

$$ad + b = cb + d$$

Answer: The functions $f \circ g$ and $g \circ f$ are equal if and only if $ad + b = cb + d$.

c: Compute the following double sum $\sum_{i=1}^2 \sum_{j=0}^4 (3i \times 4j)$.

Solution: We evaluate the inner sum first, treating i as a constant:

$$S_i = \sum_{j=0}^4 (3i \times 4j) = \sum_{j=0}^4 (12ij)$$

Factor out the terms depending on i :

$$S_i = 12i \sum_{j=0}^4 j$$

The sum of j from 0 to 4 is $0 + 1 + 2 + 3 + 4 = 10$.

$$S_i = 12i \times 10 = 120i$$

Now, substitute this result back into the outer sum:

$$\begin{aligned}\sum_{i=1}^2 S_i &= \sum_{i=1}^2 (120i) \\&= 120(1) + 120(2) \\&= 120 + 240 = 360\end{aligned}$$

Answer: The computed value of the double sum is 360.

d: Let $A = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 1 & 1 & 0 \end{pmatrix}$, **find** A^n **for all positive integers** n .

Solution: We need to compute the first few powers of the matrix A .

1. A^1

$$A^1 = A = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 1 & 1 & 0 \end{pmatrix}$$

2. A^2

$$A^2 = A \cdot A = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 1 & 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 1 & 1 & 0 \end{pmatrix}$$
$$A^2 = \begin{pmatrix} (0+0+1) & (0+0+1) & (0+0+0) \\ (0+0+0) & (0+0+0) & (1+0+0) \\ (0+1+0) & (0+0+0) & (1+0+0) \end{pmatrix} = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 1 \end{pmatrix}$$

3. A^3

$$A^3 = A^2 \cdot A = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 1 & 1 & 0 \end{pmatrix}$$
$$A^3 = \begin{pmatrix} (0+1+0) & (0+0+0) & (1+0+0) \\ (0+0+1) & (0+0+1) & (0+0+0) \\ (0+0+1) & (0+0+1) & (1+0+0) \end{pmatrix} = \begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix}$$

4. A^4

$$A^4 = A^3 \cdot A = \begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 1 & 1 & 0 \end{pmatrix}$$
$$A^4 = \begin{pmatrix} (0+0+1) & (0+0+1) & (1+0+0) \\ (0+1+0) & (0+0+0) & (1+0+0) \\ (0+1+1) & (0+0+1) & (1+0+0) \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \\ 2 & 1 & 1 \end{pmatrix}$$

The pattern is not simple or cyclic. A general closed form for A^n would require matrix diagonalization, which is a complex process.

Answer: The first four powers are:

$$A^1 = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 1 & 1 & 0 \end{pmatrix}, \quad A^2 = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 1 \end{pmatrix}$$

$$A^3 = \begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix}, \quad A^4 = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \\ 2 & 1 & 1 \end{pmatrix}$$

SOLUTION TO QUESTION 3

a: What is the power set of the empty set? What is the power set of the set $\{\emptyset\}$?

1. Power Set of the Empty Set (\emptyset)

Solution: The power set $P(A)$ of a set A is the set of all subsets of A . The only subset of the empty set \emptyset is the empty set itself.

$$P(\emptyset) = \{\emptyset\}$$

2. Power Set of the set $\{\emptyset\}$

Solution: Let $A = \{\emptyset\}$. The set A contains one element. The subsets of A are:

1. The empty set, \emptyset .
2. The set itself, $\{\emptyset\}$.

$$P(\{\emptyset\}) = \{\emptyset, \{\emptyset\}\}$$

b: List all the steps used by Bubble Sort to arrange the list 1, 8, 12, 9, 11, 2, 14, 5 in ascending order.

Solution: Bubble Sort compares adjacent elements and swaps them if they are in the wrong order.

Initial List: [1, 8, 12, 9, 11, 2, 14, 5, 10, 4]

Pass 1: (Largest element, 14, moves to end)

- [1, 8, 12, 9, 11, 2, 14, 5, 10, 4] (No swap)
- [1, 8, 12, 9, 11, 2, 14, 5, 10, 4] (No swap)
- [1, 8, **9, 12**, 11, 2, 14, 5, 10, 4] (Swap: $12 > 9$)
- [1, 8, 9, **11, 12**, 2, 14, 5, 10, 4] (Swap: $12 > 11$)
- [1, 8, 9, 11, **2, 12**, 14, 5, 10, 4] (Swap: $12 > 2$)
- [1, 8, 9, 11, 2, 12, 14, 5, 10, 4] (No swap)
- [1, 8, 9, 11, 2, 12, **5, 14**, 10, 4] (Swap: $14 > 5$)
- [1, 8, 9, 11, 2, 12, 5, **10, 14**, 4] (Swap: $14 > 10$)

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- [1, 8, 9, 11, 2, 12, 5, 10, **4, 14**] (Swap: 14 > 4)

End of Pass 1: [1, 8, 9, 11, 2, 12, 5, 10, 4, 14]

Pass 2:

- [1, 8, 9, **2, 11**, 12, 5, 10, 4, 14] (Swap: 11 > 2)
- [1, 8, 9, 2, **11, 12**, 5, 10, 4, 14] (No swap)
- [1, 8, 9, 2, 11, **5, 12**, 10, 4, 14] (Swap: 12 > 5)
- [1, 8, 9, 2, 11, 5, **10, 12**, 4, 14] (Swap: 12 > 10)
- [1, 8, 9, 2, 11, 5, 10, **4, 12**, 14] (Swap: 12 > 4)

End of Pass 2: [1, 8, 9, 2, 11, 5, 10, 4, 12, 14]

Pass 3:

- [1, 8, **2, 9**, 11, 5, 10, 4, 12, 14] (Swap: 9 > 2)
- [1, 8, 2, 9, **5, 11**, 10, 4, 12, 14] (Swap: 11 > 5)
- [1, 8, 2, 9, 5, **10, 11**, 4, 12, 14] (Swap: 11 > 10)
- [1, 8, 2, 9, 5, 10, **4, 11**, 12, 14] (Swap: 11 > 4)

End of Pass 3: [1, 8, 2, 9, 5, 10, 4, 11, 12, 14]

Pass 4:

- [1, **2, 8**, 9, 5, 10, 4, 11, 12, 14] (Swap: 8 > 2)
- [1, 2, 8, **5, 9**, 10, 4, 11, 12, 14] (Swap: 9 > 5)
- [1, 2, 8, 5, 9, **4, 10**, 11, 12, 14] (Swap: 10 > 4)

End of Pass 4: [1, 2, 8, 5, 9, 4, 10, 11, 12, 14]

Pass 5:

- [1, 2, **5, 8**, 9, 4, 10, 11, 12, 14] (Swap: 8 > 5)
- [1, 2, 5, 8, **4, 9**, 10, 11, 12, 14] (Swap: 9 > 4)

End of Pass 5: [1, 2, 5, 8, 4, 9, 10, 11, 12, 14]

Pass 6:

- [1, 2, 5, **4, 8**, 9, 10, 11, 12, 14] (Swap: 8 > 4)

End of Pass 6: [1, 2, 5, 4, 8, 9, 10, 11, 12, 14]

Pass 7:

- [1, 2, **4, 5**, 8, 9, 10, 11, 12, 14] (Swap: 5 > 4)

End of Pass 7: [1, 2, 4, 5, 8, 9, 10, 11, 12, 14]

Pass 8:

- (No swaps occur during this entire pass)

End of Pass 8: [1, 2, 4, 5, 8, 9, 10, 11, 12, 14] (Sorted)

c: Given $f(x) = 5x + 3$ and $g(x) = 6x + 4$. What is the composition of f and g , and g and f .

Solution:

1. Composition of f and g ($f \circ g$)

$$f \circ g(x) = f(g(x))$$

Substitute $g(x) = 6x + 4$ into $f(x)$:

$$\begin{aligned} f \circ g(x) &= f(6x + 4) \\ &= 5(6x + 4) + 3 \\ &= 30x + 20 + 3 \\ &= 30x + 23 \end{aligned}$$

2. Composition of g and f ($g \circ f$)

$$g \circ f(x) = g(f(x))$$

Substitute $f(x) = 5x + 3$ into $g(x)$:

$$\begin{aligned} g \circ f(x) &= g(5x + 3) \\ &= 6(5x + 3) + 4 \\ &= 30x + 18 + 4 \\ &= 30x + 22 \end{aligned}$$

Answer: $f \circ g(x) = 30x + 23$ and $g \circ f(x) = 30x + 22$.

d: Give an explicit formula for a function from the set of all integers to the set of positive integers that is onto but is not one-to-one. Also prove it with a proper example.

Solution: Let \mathbb{Z} be the set of all integers and \mathbb{Z}^+ be the set of positive integers $\{1, 2, 3, \dots\}$. We seek a function $f : \mathbb{Z} \rightarrow \mathbb{Z}^+$.

1. Explicit Formula

$$f(x) = |x| + 1$$

This function maps all integers to positive integers, as $|x| \geq 0$, so $|x| + 1 \geq 1$.

2. Proof that $f(x)$ is Onto (Surjective)

We must show that for any $y \in \mathbb{Z}^+$, there is an $x \in \mathbb{Z}$ such that $f(x) = y$. Let y be an arbitrary element of \mathbb{Z}^+ . This means $y \geq 1$. We need to find x such that $f(x) = y$, or $|x| + 1 = y$. This implies $|x| = y - 1$. Since $y \geq 1$, the value $k = y - 1$ is an integer ≥ 0 . We can choose $x = k = y - 1$. Since y is an integer, x is also an integer. Let's check: $f(x) = f(y-1) = |y-1| + 1$. Since $y \geq 1$, $y-1 \geq 0$, so $|y-1| = y-1$. $f(y-1) = (y-1) + 1 = y$. Since we found a pre-image $x = (y - 1)$ for every y in the codomain, the function is **onto**.

3. Proof that $f(x)$ is Not One-to-One (Injective)

We must find two distinct integers $x_1 \neq x_2$ such that $f(x_1) = f(x_2)$. **Example:** Let $x_1 = 2$ and $x_2 = -2$.

$$f(x_1) = f(2) = |2| + 1 = 2 + 1 = 3$$

$$f(x_2) = f(-2) = |-2| + 1 = 2 + 1 = 3$$

Since $x_1 \neq x_2$ but $f(x_1) = f(x_2)$, the function is **not one-to-one**.