BANGLADESH ARMY INTERNATIONAL UNIVERSITY OF SCIENCE AND TECHNOLOGY (BAIUST), CUMILLA

Mid Term Examination, Fall 2024

Department of Computer Science and Engineering (CSE)

Level-1, Term-1

Course Code: CSE 111 **Full Marks:** 90
Course Title: Mathematics-1
Credit Hour: 03 **Time:** 1 hr. 30 mins

ANSWER SHEET

Examiner's Note: Answer any three (03) of the following four (04) questions including Q.No.-1.
Student's Note: All four questions are answered below for reference and study purposes.

Question-01

a. Discuss the continuity of the function $f(x)$ at $x = \frac{1}{2}$

The function $f(x)$ is defined as:

$$f(x) = \begin{cases} \frac{1}{2} - x & \text{when } x < \frac{1}{2} \\ \frac{1}{2} & \text{when } x = \frac{1}{2} \\ \frac{3}{2} - x & \text{when } x > \frac{1}{2} \end{cases}$$

For a function to be continuous at a point $x = a$, the following three conditions must be met:

1. $f(a)$ must be defined.
2. $\lim_{x \to a} f(x)$ must exist, meaning $\lim_{x \to a^-} f(x) = \lim_{x \to a^+} f(x)$.
3. $\lim_{x \to a} f(x) = f(a)$.

1. Value of the function at $x = \frac{1}{2}$

From the definition:

$$f\left(\frac{1}{2}\right) = \frac{1}{2}$$

Condition 1 is satisfied.

2. Left-hand limit (LHL) at $x = \frac{1}{2}$

For $x < \frac{1}{2}$, $f(x) = \frac{1}{2} - x$:

$$\text{LHL} = \lim_{x \to \frac{1}{2}^-} f(x) = \lim_{x \to \frac{1}{2}^-} \left(\frac{1}{2} - x\right) = \frac{1}{2} - \frac{1}{2} = 0$$

3. Right-hand limit (RHL) at $x = \frac{1}{2}$

For $x > \frac{1}{2}$, $f(x) = \frac{3}{2} - x$:

$$\text{RHL} = \lim_{x \to \frac{1}{2}^+} f(x) = \lim_{x \to \frac{1}{2}^+} \left(\frac{3}{2} - x\right) = \frac{3}{2} - \frac{1}{2} = \frac{2}{2} = 1$$

Conclusion on Continuity

Since the Left-hand limit and the Right-hand limit are not equal:

$$\text{LHL} \neq \text{RHL} \implies 0 \neq 1$$

Therefore, $\lim_{x \to \frac{1}{2}} f(x)$ **does not exist**. **Condition 2 is not satisfied.** Thus, the function $f(x)$ is **discontinuous** at $x = \frac{1}{2}$.

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b: Compute the value of $\lim_{y \to 0} \frac{y + \sin y}{y}$

We are asked to compute the limit:

$$L = \lim_{y \to 0} \frac{y + \sin y}{y}$$

We can split the fraction:

$$L = \lim_{y \to 0} \left(\frac{y}{y} + \frac{\sin y}{y}\right)$$

$$L = \lim_{y \to 0} \left(1 + \frac{\sin y}{y}\right)$$

We know the standard trigonometric limit: $\lim_{\theta \to 0} \frac{\sin\theta}{\theta} = 1$. Applying the limit properties:

$$L = \lim_{y \to 0} 1 + \lim_{y \to 0} \frac{\sin y}{y}$$
$$L = 1 + 1$$
$$L = 2$$

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c: Explain the differential coefficient of the following with respect to x: $2\sin^2 2x + 5\log(2x) + e^x$

We need to find the derivative of $y = 2\sin^2 2x + 5\log(2x) + e^x$ with respect to x.

$$\frac{dy}{dx} = \frac{d}{dx}\left(2\sin^2 2x\right) + \frac{d}{dx}\left(5\log(2x)\right) + \frac{d}{dx}\left(e^x\right)$$

Term 1: $\frac{d}{dx}\left(2\sin^2 2x\right)$

Using the Chain Rule, $2\sin^2 2x = 2(\sin 2x)^2$.

$$\begin{aligned}\frac{d}{dx}\left(2(\sin 2x)^2\right) &= 2 \cdot 2(\sin 2x)^{2-1} \cdot \frac{d}{dx}(\sin 2x) \\ &= 4\sin 2x \cdot (\cos 2x) \cdot \frac{d}{dx}(2x) \\ &= 4\sin 2x \cos 2x \cdot 2 \\ &= 8\sin 2x \cos 2x\end{aligned}$$

Using the identity $\sin 2\theta = 2\sin\theta\cos\theta$:

$$8\sin 2x \cos 2x = 4(2\sin 2x \cos 2x) = 4\sin(2 \cdot 2x) = 4\sin 4x$$

So, $\frac{d}{dx}\left(2\sin^2 2x\right) = 4\sin 4x$.

Term 2: $\frac{d}{dx}\left(5\log(2x)\right)$

Using the Chain Rule and $\frac{d}{dx}(\log u) = \frac{1}{u}\frac{du}{dx}$:

$$\begin{aligned}\frac{d}{dx}\left(5\log(2x)\right) &= 5 \cdot \frac{1}{2x} \cdot \frac{d}{dx}(2x) \\ &= 5 \cdot \frac{1}{2x} \cdot 2 \\ &= \frac{5}{x}\end{aligned}$$

Term 3: $\frac{d}{dx}(e^x)$

The derivative of e^x is e^x.

$$\frac{d}{dx}(e^x) = e^x$$

Final Result

Combining the derivatives of the three terms:

$$\frac{dy}{dx} = 4\sin 4x + \frac{5}{x} + e^x$$

Question-02

a: Write the solution of the integral $\int \sin x \log(\sec x + \tan x)\,dx$ by parts.

We will use the integration by parts formula:

$$\int u\,dv = uv - \int v\,du$$

We choose $u = \log(\sec x + \tan x)$ and $dv = \sin x\,dx$.

1. Find du and v

- **Finding du:**

$$u = \log(\sec x + \tan x)$$

  The derivative of $\log(\sec x + \tan x)$ is $\sec x$.

$$\frac{du}{dx} = \frac{1}{\sec x + \tan x} \cdot \frac{d}{dx}(\sec x + \tan x)$$

$$\frac{du}{dx} = \frac{1}{\sec x + \tan x} \cdot (\sec x \tan x + \sec^2 x)$$

$$\frac{du}{dx} = \frac{\sec x(\tan x + \sec x)}{\sec x + \tan x}$$

$$\frac{du}{dx} = \sec x$$

  Therefore, $du = \sec x\,dx$.

- **Finding v:**

$$dv = \sin x\,dx$$

$$v = \int \sin x\,dx = -\cos x$$

2. Apply the Integration by Parts formula

$$\int \sin x \log(\sec x + \tan x)\, dx = uv - \int v\, du$$

$$\begin{aligned}
&= \log(\sec x + \tan x)(-\cos x) - \int (-\cos x)(\sec x)\, dx \\
&= -\cos x \log(\sec x + \tan x) + \int \cos x \cdot \frac{1}{\cos x}\, dx \\
&= -\cos x \log(\sec x + \tan x) + \int 1\, dx \\
&= -\cos x \log(\sec x + \tan x) + x + C
\end{aligned}$$

The solution is:

$$\int \sin x \log(\sec x + \tan x)\, dx = x - \cos x \log(\sec x + \tan x) + C$$

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b: Compute the integral $\int \sin^2 x \cos 2x\, dx$.

We will use trigonometric identities to simplify the integrand.

- Use the identity: $\sin^2 x = \frac{1-\cos 2x}{2}$.
- Use the identity: $\cos^2 2x = \frac{1+\cos 4x}{2}$.

Substitute the first identity:

$$I = \int \sin^2 x \cos 2x\, dx = \int \left(\frac{1-\cos 2x}{2}\right) \cos 2x\, dx$$

$$I = \frac{1}{2} \int (\cos 2x - \cos^2 2x)\, dx$$

$$I = \frac{1}{2} \left[\int \cos 2x\, dx - \int \cos^2 2x\, dx\right]$$

Now, we calculate the two integrals separately:

1. $\int \cos 2x\, dx = \frac{\sin 2x}{2}$
2. For $\int \cos^2 2x\, dx$, we use the second identity:

$$\int \cos^2 2x\, dx = \int \left(\frac{1+\cos(2 \cdot 2x)}{2}\right) dx = \frac{1}{2} \int (1 + \cos 4x)\, dx$$

$$= \frac{1}{2} \left[x + \frac{\sin 4x}{4}\right]$$

Substitute these results back into the expression for I:

$$I = \frac{1}{2}\left[\left(\frac{\sin 2x}{2}\right) - \frac{1}{2}\left(x + \frac{\sin 4x}{4}\right)\right] + C$$
$$I = \frac{1}{2}\left[\frac{\sin 2x}{2} - \frac{x}{2} - \frac{\sin 4x}{8}\right] + C$$
$$I = \frac{\sin 2x}{4} - \frac{x}{4} - \frac{\sin 4x}{16} + C$$

The computed integral is:

$$\int \sin^2 x \cos 2x \, dx = \frac{\sin 2x}{4} - \frac{x}{4} - \frac{\sin 4x}{16} + C$$

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c: Indicate the solution of the integral $\int_0^a \frac{dx}{(a^2+x^2)^{3/2}}$

We use the trigonometric substitution method. Let $x = a\tan\theta$. Then $dx = a\sec^2\theta \, d\theta$.

1. Change the limits of integration

- When $x = 0$: $0 = a\tan\theta \implies \tan\theta = 0 \implies \theta = 0$.
- When $x = a$: $a = a\tan\theta \implies \tan\theta = 1 \implies \theta = \frac{\pi}{4}$.

The new limits are from $\theta = 0$ to $\theta = \frac{\pi}{4}$.

2. Simplify the denominator

$$a^2 + x^2 = a^2 + (a\tan\theta)^2 = a^2 + a^2\tan^2\theta = a^2(1 + \tan^2\theta)$$

Since $1 + \tan^2\theta = \sec^2\theta$:

$$a^2 + x^2 = a^2\sec^2\theta$$

Now, substitute this into the power $\frac{3}{2}$:

$$(a^2 + x^2)^{3/2} = (a^2\sec^2\theta)^{3/2} = a^{2\cdot\frac{3}{2}}(\sec^2\theta)^{\frac{3}{2}} = a^3\sec^3\theta$$

3. Substitute into the integral

$$I = \int_0^{\pi/4} \frac{a \sec^2\theta \, d\theta}{a^3 \sec^3\theta}$$
$$I = \int_0^{\pi/4} \frac{1}{a^2} \frac{\sec^2\theta}{\sec^3\theta} \, d\theta$$
$$I = \frac{1}{a^2} \int_0^{\pi/4} \frac{1}{\sec\theta} \, d\theta$$
$$I = \frac{1}{a^2} \int_0^{\pi/4} \cos\theta \, d\theta$$

4. Evaluate the definite integral

$$I = \frac{1}{a^2} [\sin\theta]_0^{\pi/4}$$
$$I = \frac{1}{a^2} \left[\sin\left(\frac{\pi}{4}\right) - \sin(0)\right]$$
$$I = \frac{1}{a^2} \left[\frac{1}{\sqrt{2}} - 0\right]$$
$$I = \frac{1}{a^2\sqrt{2}}$$

The solution of the integral is:

$$\int_0^a \frac{dx}{(a^2 + x^2)^{3/2}} = \frac{1}{a^2\sqrt{2}}$$

Question-03

a: Identify the solution of the integral $\int \frac{e^x(1+x)}{\cos^2(xe^x)} \, dx$ by the method of substitution.

We use the method of substitution. We observe that the numerator $e^x(1+x)\,dx$ is related to the derivative of the argument inside the cosine function, xe^x.

1. Substitution

Let $u = xe^x$.

2. Find du

Differentiate u with respect to x using the product rule:

$$\frac{du}{dx} = \frac{d}{dx}(x) \cdot e^x + x \cdot \frac{d}{dx}(e^x)$$

$$\frac{du}{dx} = 1 \cdot e^x + x \cdot e^x = e^x(1+x)$$

Therefore, the differential is:

$$du = e^x(1+x)\,dx$$

3. Substitute into the Integral

Substituting u and du into the original integral:

$$I = \int \frac{e^x(1+x)}{\cos^2(xe^x)}\,dx = \int \frac{du}{\cos^2 u}$$

Since $\frac{1}{\cos u} = \sec u$:

$$I = \int \sec^2 u\,du$$

4. Evaluate the Integral

The standard integral of $\sec^2 u$ is $\tan u$:

$$I = \tan u + C$$

5. Back-substitute

Substitute $u = xe^x$ back into the result:

$$I = \tan(xe^x) + C$$

The solution is:

$$\int \frac{e^x(1+x)}{\cos^2(xe^x)}\,dx = \tan(xe^x) + C$$

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b: Give a solution of $\int \frac{\cos x - \sin x}{\cos x + \sin x}(2 + 2\sin 2x)\,dx$.

Let the integral be I.

$$I = \int \frac{\cos x - \sin x}{\cos x + \sin x}(2 + 2\sin 2x)\,dx$$

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1. Simplify the Term $2+2\sin 2x$

We use the trigonometric identity $\sin 2x = 2\sin x\cos x$.

$$\begin{aligned} 2+2\sin 2x &= 2(1+\sin 2x) \\ &= 2(1+2\sin x\cos x) \end{aligned}$$

We also recognize that the identity $1=\cos^2 x+\sin^2 x$.

$$\begin{aligned} 2+2\sin 2x &= 2(\cos^2 x+\sin^2 x+2\sin x\cos x) \\ &= 2(\cos x+\sin x)^2 \end{aligned}$$

2. Substitute the Simplified Term back into the Integral

$$I=\int \frac{\cos x-\sin x}{\cos x+\sin x}\cdot 2(\cos x+\sin x)^2\,dx$$

Cancel one factor of $(\cos x+\sin x)$:

$$I=2\int(\cos x-\sin x)(\cos x+\sin x)\,dx$$

3. Simplify using Difference of Squares

Using the identity $(a-b)(a+b)=a^2-b^2$:

$$(\cos x-\sin x)(\cos x+\sin x)=\cos^2 x-\sin^2 x$$

We know the double angle identity $\cos^2 x-\sin^2 x=\cos 2x$.

$$I=2\int\cos 2x\,dx$$

4. Evaluate the Final Integral

$$I=2\cdot\left(\frac{\sin 2x}{2}\right)+C$$

$$I=\sin 2x+C$$

The solution is:

$$\int\frac{\cos x-\sin x}{\cos x+\sin x}(2+2\sin 2x)\,dx=\sin 2x+C$$

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c: Explain the integral $\int_0^{\pi/2} x^2 \sin x \, dx$.

We use the method of **Integration by Parts** repeatedly (tabular method or successive application). The general formula is $\int u \, dv = uv - \int v \, du$. Since the process involves two applications, we can use the pattern:

$$\int u \, dv = uv_1 - u'v_2 + u''v_3 - \dots$$

Here, we let u be the polynomial part and dv be the trigonometric part.

1. Choose u and dv, and find successive derivatives and integrals

Derivatives of $u = x^2$ ($u^{(n)}$)	**Integrals of $dv = \sin x \, dx$ (v_{n+1})**
$u = x^2$	$v_1 = \int \sin x \, dx = -\cos x$
$u' = 2x$	$v_2 = \int (-\cos x) \, dx = -\sin x$
$u'' = 2$	$v_3 = \int (-\sin x) \, dx = \cos x$
$u''' = 0$	$v_4 = \dots$

2. Apply the Pattern

$$\begin{aligned}\int x^2 \sin x \, dx &= uv_1 - u'v_2 + u''v_3 \\ &= (x^2)(-\cos x) - (2x)(-\sin x) + (2)(\cos x) \\ &= -x^2 \cos x + 2x \sin x + 2 \cos x\end{aligned}$$

3. Evaluate the Definite Integral

$$I = \left[-x^2 \cos x + 2x \sin x + 2 \cos x\right]_0^{\pi/2}$$

- **Upper Limit (x $= \pi/2$):**

$$\begin{aligned}F\left(\frac{\pi}{2}\right) &= -\left(\frac{\pi}{2}\right)^2 \cos\left(\frac{\pi}{2}\right) + 2\left(\frac{\pi}{2}\right) \sin\left(\frac{\pi}{2}\right) + 2\cos\left(\frac{\pi}{2}\right) \\ &= -\frac{\pi^2}{4}(0) + \pi(1) + 2(0) \\ &= 0 + \pi + 0 = \pi\end{aligned}$$

- **Lower Limit (x $= 0$):**

$$\begin{aligned}F(0) &= -(0)^2 \cos(0) + 2(0)\sin(0) + 2\cos(0) \\ &= -0(1) + 0(0) + 2(1) \\ &= 2\end{aligned}$$

4. Final Result

$$I = F\left(\frac{\pi}{2}\right) - F(0) = \pi - 2$$

The solution is:

$$\int_0^{\pi/2} x^2 \sin x \, dx = \pi - 2$$