Lecture 14: Game Theory

Game Theory

- Consider the following examples:
  - parlor games,
  - military battles,
  - political campaigns,
  - advertising and marketing campaigns by competing business firms.
- A basic feature in many of these situations is that the final outcome depends primarily upon the combination of strategies selected by the adversaries.
- Game theory is a mathematical theory that deals with the general features of competitive situations like these in a formal, abstract way.

Game Theory

- Our focus in this lecture is on the simplest case, called **two-person, zero-sum games.**
- These games involve only two adversaries or players (who may be armies, teams, firms, and so on).
- They are called zero-sum games because one player wins whatever the other one loses, so that the sum of their net winnings is zero.

The Formulation of Two-person, Zero-sum Games

- Consider the game called odds and evens. This game consists simply of each player simultaneously showing either one finger or two fingers. If the number of fingers matches, so that the total number for both players is even, then the player taking evens (say, player 1) wins the bet (say, $1) from the player taking odds (player 2). If the number does not match, player 1 pays $1 to player 2. Thus, each player has two strategies: to show either one finger or two fingers.

The Formulation of Two-person, Zero-sum Games

In general, a two-person game is characterized by

1. The strategies of player 1
2. The strategies of player 2
3. The payoff table

Payoff table for the odds and evens game

		Player 2	
Strategy		1	2
Player 1	1	1	−1
	2	−1	1

The Formulation of Two-person, Zero-sum Games

- Before the game begins, each player knows the strategies she or he has available, the ones the opponent has available, and the payoff table.
- The actual play of the game consists of each player simultaneously choosing a strategy without knowing the opponent's choice.

Strategy

- A strategy may involve only a simple action, such as showing a certain number of fingers in the odds and evens game.
- On the other hand, in more complicated games involving a series of moves, a **strategy is a predetermined rule that specifies completely** how one intends to respond to each possible circumstance at each stage of the game.
- For example, a strategy for one side in chess would indicate how to make the next move for *every possible position on the board, so the total number of possible strategies would be* astronomical.

The payoff table

- The **payoff table shows the gain (positive or negative) for player 1 that would result** from each combination of strategies for the two players.
- It is given only for player 1 because the table for player 2 is just the negative of this one, due to the zero-sum nature of the game.
- The entries in the payoff table may be in any units desired, such as dollars, provided that they accurately represent the *utility to player 1 of the corresponding outcome.*

Assumptions

- A primary objective of game theory is the development of rational criteria for selecting a strategy.
- Two key assumptions are made:

  1. Both players are rational.

  2. Both players choose their strategies solely to promote their own welfare (no compassion for the opponent).

Game Theory vs. Decision Analysis

- Game theory contrasts with *decision analysis,* where the *assumption* is that the decision maker is playing a game with a passive opponent—nature—which chooses its strategies in some random fashion.

A Prototype Example

- Two politicians are running against each other for the U.S. Senate. Campaign plans must now be made for the final 2 days, which are expected to be crucial because of the closeness of the race. Therefore, both politicians want to spend these days campaigning in two key cities, Bigtown and Megalopolis. To avoid wasting campaign time, they plan to travel at night and spend either 1 full day in each city or 2 full days in just one of the cities. However, since the necessary arrangements must be made in advance, neither politician will learn his (or her) opponent's campaign schedule until after he has finalized his own. Therefore, each politician has asked his campaign manager in each of these cities to assess what the impact would be (in terms of votes won or lost) from the various possible combinations of days spent there by himself and by his opponent. He then wishes to use this information to choose his best strategy on how to use these 2 days.

Formulation as a Two-Person, Zero-Sum Game

- Each player has the following strategies:

  Strategy 1 = spend 1 day in each city.
  Strategy 2 = spend both days in Bigtown.
  Strategy 3 = spend both days in Megalopolis.

- Payoff table:

		Total Net Votes Won by Politician 1 (in Units of 1,000 Votes)		
		Politician 2		
	Strategy	1	2	3
	1			
Politician 1	2			
	3			

Variation 1 of the Example

- Given that table below is the payoff table for player 1 (politician 1), which strategy should each player select?

		Player 2		
Strategy		1	2	3
Player 1	1	1	2	4
	2	1	0	5
	3	0	1	−1

- This situation is a rather special one, where the answer can be obtained just by applying the concept of **dominated strategies** to rule out a succession of inferior strategies until only one choice remains.

Dominated Strategies

- A strategy is **dominated by a second strategy if the second strategy is *always at least as*** *good (and sometimes better) regardless of what the opponent does. A dominated strategy* can be eliminated immediately from further consideration.

		Player 2		
Strategy		1	2	3
Player 1	1	1	2	4
	2	1	0	5
	3	0	1	−1

- At the outset, the table includes no dominated strategies for player 2.

Dominated Strategies

- For player 1, strategy 3 is dominated by strategy 1 because the latter has larger payoffs (1 > 0, 2 > 1, 4 > -1) regardless of what player 2 does.
- Eliminating strategy 3 from further consideration yields the following reduced payoff table:

	1	2	3
1	1	2	4
2	1	0	5

- Because both players are assumed to be rational, player 2 also can deduce that player 1 has only these two strategies remaining under consideration.

Dominated Strategies

	1	2
1	1	2
2	1	0

	1	2
1	1	2

	1
1	1

- Consequently, both players should select their strategy 1.
- Player 1 then will receive a payoff of 1 from player 2 (that is, politician 1 will gain 1,000 votes from politician 2).
- In general, the payoff to player 1 when both players play optimally is referred to as the **value of the game.**
- **A game that has a value of 0 is said to be a fair game.**

Variation 2 of the Example

Strategy		Player 2: 1	2	3
Player 1	1	−3	−2	6
	2	2	0	2
	3	5	−2	−4

The minimax criterion

- In terms of the payoff table, it implies that *player 1 should select the strategy whose minimum payoff is largest, whereas player 2 should choose the one whose maximum payoff to player 1 is the smallest.*

Strategy		Player 2: 1	2	3	Minimum
Player 1	1	−3	−2	6	−3
	2	2	0	2	0 ← Maximin value
	3	5	−2	−4	−4
Maximum:		5	0 ↑ Minimax value	6	

- Notice the interesting fact that the same entry in this payoff table yields both the maximin and minimax values.
- The position of any such entry is called a **saddle point.**
- Since this is a **stable solution (also called an *equilibrium* solution), *players 1 and 2 should exclusively*** use their maximin and minimax strategies, respectively.

Variation 3 of the Example

Strategy		Player 2: 1	2	3
Player 1	1	0	−2	2
	2	5	4	−3
	3	2	3	−4

Strategy		Player 2: 1	2	3	Minimum
Player 1	1	0	−2	2	−2 ← Maximin value
	2	5	4	−3	−3
	3	2	3	−4	−4
Maximum:		5	4	2 ↑ Minimax value	

Notice that the maximin value (-2) and the minimax value (2) do not coincide in this case. The result is that there is *no saddle point.*

Variation 3 of the Example

- The originally suggested solution (player 1 to play strategy 1 and player 2 to play strategy 3) is an **unstable solution, so it is necessary to develop a more satisfactory** solution.
- An essential feature of a rational plan for playing a game such as this one is that **neither player should be able to deduce which strategy the other will use**.
- It is necessary to choose among alternative acceptable strategies on some kind of random basis.

Games with Mixed Strategies

- Whenever a game does not possess a saddle point, game theory advises each player to assign a probability distribution over her set of strategies.

x_i = probability that player 1 will use strategy i ($i = 1, 2, \dots, m$),
y_j = probability that player 2 will use strategy j ($j = 1, 2, \dots, n$),

- These plans ($x_1, x_2, \dots, x_m$) and ($y_1, y_2, \dots, y_n$) are usually referred to as mixed strategies, and the original strategies are then called pure strategies.

Games with Mixed Strategies

Strategy		Player 2: 1	2	3
Player 1	1	0	−2	2
	2	5	4	−3
	3	2	3	−4
	Maximum:	5	4	2

$(x_1, x_2, x_3) = (\frac{1}{2}, \frac{1}{2}, 0)$ $(y_1, y_2, y_3) = (0, \frac{1}{2}, \frac{1}{2})$

Games with Mixed Strategies

- Although no completely satisfactory measure of performance is available for evaluating mixed strategies, a very useful one is the expected payoff.
- By applying the probability theory definition of expected value, this quantity is

$$\text{Expected payoff for player 1} = \sum_{i=1}^{m} \sum_{j=1}^{n} p_{ij} x_i y_j.$$

- where p_{ij} is the payoff if player 1 uses pure strategy i and player 2 uses pure strategy j.

Games with Mixed Strategies

- In this context, the **minimax criterion says that a given player should select the mixed strategy that minimizes the maximum** expected loss to himself.
- Equivalently, when we focus on payoffs (player 1) rather than losses (player 2), this criterion says to maximin instead, i.e., maximize the minimum expected payoff to the player.
- Player 1: the one that provides the guarantee (minimum expected payoff) that is best (maximal).
- Player 2: the one that provides the best guarantee, where best now means minimal and guarantee refers to the maximum expected loss.

Games with Mixed Strategies

- Player 1: The value of this best guarantee is the maximin value, denoted by $\underline{v}$.
- Player 2: This best guarantee is the minimax value, denoted by $\overline{v}$.
- For games with mixed strategies, it is necessary that

$$\underline{v} = \overline{v}$$

for the optimal solution to be *stable*.

Games with Mixed Strategies

Minimax theorem: If mixed strategies are allowed, the pair of mixed strategies that is optimal according to the minimax criterion provides a *stable solution* with $\underline{v} = \overline{v} = v$ (the value of the game), so that neither player can do better by unilaterally changing her or his strategy.

Graphical Solution Procedure

- Graphical procedure may be used whenever one of the players has only two (undominated) pure strategies.

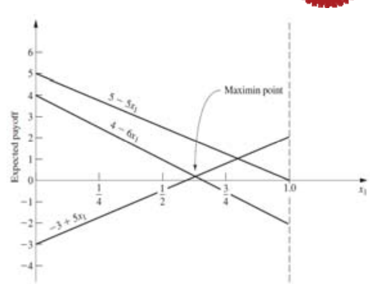
		Probability	Player 2 y_1	y_2	y_3
	Probability	**Pure Strategy**	**1**	**2**	**3**
Player 1	x_1	1	0	−2	2
	$1 - x_1$	2	5	4	−3

- For each of the pure strategies available to player 2, the expected payoff for player 1 will be

(y_1, y_2, y_3)	Expected Payoff
(1, 0, 0)	$0x_1 + 5(1 - x_1) = 5 - 5x_1$
(0, 1, 0)	$-2x_1 + 4(1 - x_1) = 4 - 6x_1$
(0, 0, 1)	$2x_1 - 3(1 - x_1) = -3 + 5x_1$

Graphical Solution Procedure

(y_1, y_2, y_3)	Expected Payoff
(1, 0, 0)	$0x_1 + 5(1 - x_1) = 5 - 5x_1$
(0, 1, 0)	$-2x_1 + 4(1 - x_1) = 4 - 6x_1$
(0, 0, 1)	$2x_1 - 3(1 - x_1) = -3 + 5x_1$



Graphical Solution Procedure

- For any given values of x_1 and (y_1, y_2, y_3), the expected payoff will be the appropriate weighted average of the corresponding points on these three lines.

Expected payoff for player 1 $= y_1(5 - 5x_1) + y_2(4 - 6x_1) + y_3(-3 + 5x_1)$.

$$\underline{v} = v = \max_{0 \le x_1 \le 1} \{\min\{-3 + 5x_1, 4 - 6x_1\}\}.$$

which yields $x_1 = \frac{7}{11}$. Thus, $(x_1, x_2) = (\frac{7}{11}, \frac{4}{11})$ is the *optimal mixed strategy* for player 1, and

$$\underline{v} = v = -3 + 5\left(\frac{7}{11}\right) = \frac{2}{11}$$

is the value of the game.

Graphical Solution Procedure

$$y_1^*(5 - 5x_1) + y_2^*(4 - 6x_1) + y_3^*(-3 + 5x_1) \le \bar{v} = v = \frac{2}{11}$$

for all values of x_1 ($0 \le x_1 \le 1$). Furthermore, when player 1 is playing optimally (that is, $x_1 = \frac{7}{11}$), this inequality will be an equality (by the minimax theorem), so that

$$\frac{20}{11}y_1^* + \frac{2}{11}y_2^* + \frac{2}{11}y_3^* = v = \frac{2}{11}.$$

Because (y_1, y_2, y_3) is a probability distribution, it is also known that

$$y_1^* + y_2^* + y_3^* = 1.$$

Therefore, $y_1^* = 0$ because $y_1^* > 0$ would violate the next-to-last equation; i.e., the expected payoff on the graph at $x_1 = \frac{7}{11}$ would be above the maximin point. (In general, any line that does not pass through the maximin point must be given a zero weight to avoid increasing the expected payoff above this point.)

Graphical Solution Procedure

$$y_2^*(4 - 6x_1) + y_3^*(-3 + 5x_1) \begin{cases} \le \frac{2}{11} & \text{for } 0 \le x_1 \le 1, \\ = \frac{2}{11} & \text{for } x_1 = \frac{7}{11}. \end{cases}$$

$$4y_2^* - 3y_3^* = \frac{2}{11},$$
$$-2y_2^* + 2y_3^* = \frac{2}{11},$$

which has a simultaneous solution of $y_2^* = \frac{5}{11}$ and $y_3^* = \frac{6}{11}$. Therefore, the *optimal mixed strategy* for player 2 is $(y_1, y_2, y_3) = (0, \frac{5}{11}, \frac{6}{11})$.

Graphical Solution Procedure

- If, in another problem, there should happen to be more than two lines passing through the maximin point, so that more than two of the y_j* *values can be greater than zero,* this condition would imply that there are many ties for the optimal mixed strategy for player 2.
- One such strategy can then be identified by setting all but two of these y_j* values equal to zero and solving for the remaining two in the manner just described.

LP Formulation

Minimax theorem: If mixed strategies are allowed, the pair of mixed strategies that is optimal according to the minimax criterion provides a *stable solution* with $\underline{v} = \overline{v} = v$ (the value of the game), so that neither player can do better by unilaterally changing her or his strategy.

$$\text{Expected payoff for player 1} = \sum_{i=1}^{m} \sum_{j=1}^{n} p_{ij} x_i y_j$$

and the strategy $(x_1, x_2, \ldots, x_m)$ is optimal if

$$\sum_{i=1}^{m} \sum_{j=1}^{n} p_{ij} x_i y_j \geq \underline{v} = v \quad \text{for all opposing strategies } (y_1, y_2, \ldots, y_n).$$

LP Formulation

$$\sum_{i=1}^{m} \sum_{j=1}^{n} p_{ij} x_i y_j \geq \underline{v} = v$$

$$\sum_{i=1}^{m} p_{ij} x_i \geq v \quad \text{for } j = 1, 2, \ldots, n,$$

Maximize x_{m+1},

subject to

$$\begin{aligned} p_{11}x_1 + p_{21}x_2 + \cdots + p_{m1}x_m - x_{m+1} &\geq 0 \\ p_{12}x_1 + p_{22}x_2 + \cdots + p_{m2}x_m - x_{m+1} &\geq 0 \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ p_{1n}x_1 + p_{2n}x_2 + \cdots + p_{mn}x_m - x_{m+1} &\geq 0 \\ x_1 + x_2 + \cdots + x_m &= 1 \end{aligned}$$

and

$$x_i \geq 0, \quad \text{for } i = 1, 2, \ldots, m.$$

LP Formulation

Minimize y_{n+1},

subject to

$$\begin{aligned} p_{11}y_1 + p_{12}y_2 + \cdots + p_{1n}y_n - y_{n+1} &\leq 0 \\ p_{21}y_1 + p_{22}y_2 + \cdots + p_{2n}y_n - y_{n+1} &\leq 0 \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ p_{m1}y_1 + p_{m2}y_2 + \cdots + p_{mn}y_n - y_{n+1} &\leq 0 \\ y_1 + y_2 + \cdots + y_n &= 1 \end{aligned}$$

and

$$y_j \geq 0, \quad \text{for } j = 1, 2, \ldots, n.$$

Example

			Player 2		
		Probability	y_1	y_2	y_3
	Probability	Pure Strategy	1	2	3
Player 1	x_1	1	0	−2	2
	$1 - x_1$	2	5	4	−3

Maximize x_{m+1},

subject to

$$\begin{aligned} p_{11}x_1 + p_{21}x_2 + \cdots + p_{m1}x_m - x_{m+1} &\geq 0 \\ p_{12}x_1 + p_{22}x_2 + \cdots + p_{m2}x_m - x_{m+1} &\geq 0 \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ p_{1n}x_1 + p_{2n}x_2 + \cdots + p_{mn}x_m - x_{m+1} &\geq 0 \\ x_1 + x_2 + \cdots + x_m &= 1 \end{aligned}$$

and

$$x_i \geq 0, \quad \text{for } i = 1, 2, \ldots, m.$$

Maximize x_3,

subject to

$$\begin{aligned} 5x_2 - x_3 &\geq 0 \\ -2x_1 + 4x_2 - x_3 &\geq 0 \\ 2x_1 - 3x_2 - x_3 &\geq 0 \\ x_1 + x_2 &= 1 \end{aligned}$$

and

$$x_1 \geq 0, \quad x_2 \geq 0.$$

Example

		Probability	Player 2 y_1	y_2	y_3
	Probability	Pure Strategy	1	2	3
Player 1	x_1	1	0	−2	2
	$1-x_1$	2	5	4	−3

Minimize y_{n+1},

subject to

$$p_{11}y_1 + p_{12}y_2 + \cdots + p_{1n}y_n - y_{n+1} \le 0$$
$$p_{21}y_1 + p_{22}y_2 + \cdots + p_{2n}y_n - y_{n+1} \le 0$$
$$\cdots$$
$$p_{m1}y_1 + p_{m2}y_2 + \cdots + p_{mn}y_n - y_{n+1} \le 0$$
$$y_1 + y_2 + \cdots + y_n = 1$$

and

$$y_j \ge 0, \quad \text{for } j = 1, 2, \ldots, n.$$

Minimize y_4,

subject to

$$-2y_2 + 2y_3 - y_4 \le 0$$
$$5y_1 + 4y_2 - 3y_3 - y_4 \le 0$$
$$y_1 + y_2 + y_3 = 1$$

and

$$y_1 \ge 0, \quad y_2 \ge 0, \quad y_3 \ge 0.$$

Assignment

- Problems 14.1-2, 14.1-3, 14.2-6, 14.2-7, 14.3-1, 14.4-4.