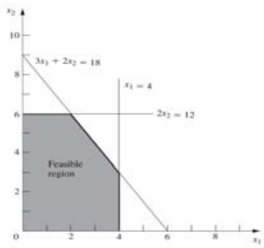
Lecture 09: Introduction to Linear Programming



Outline

- Identify:
  - Objective function
  - Constraints
  - Optimization variables
  - Feasible region
- LP Terminology
- Examples

Linear Programming (LP)

- LP has been ranked among the most important scientific advances of the mid-20th century.
- What is the nature of this remarkable tool, and what kinds of problems does it address?
- General problem of allocating limited resources among competing activities in a best possible (i.e., optimal) way.
- The adjective linear means that all the mathematical functions in this model are required to be linear functions.

Linear Programming (LP): Business Definition

- A mathematical technique designed to help operations managers plan and make decisions relative to the trade-offs necessary to allocate resources.

Linear Programming (LP) : Examples

- Scheduling school buses to *minimize the total distance traveled when carrying students.*
- Allocating police patrol units to high crime areas to *minimize response time to 911* calls.
- Scheduling tellers at banks so that needs are met during each hour of the day while *minimizing the total cost of labor.*
- Selecting the product mix in a factory to make best use of machine- and labor-hours available while *maximizing the firm's profit.*

Linear Programming (LP) : Examples

- Determining the distribution system that will *minimize total shipping cost from several* warehouses to various market locations.
- Developing a production schedule that will satisfy future demands for a firm's product and at the same time *minimize total production and inventory costs.*

Linear Programming (LP): Major Components

- Objective Function: LP problems seek to *maximize or minimize some quantity (usually profit or cost). We refer* to this property as the **objective function** of an LP problem.
- Constraints: The presence of restrictions, or **constraints,** limits the degree to which we can pursue our objective. For example, deciding how many units of each product in a firm's product line to manufacture is restricted by available labor and machinery
- Decision/Optimization variables: The variables that change at each iteration of optimization.

Prototype Example: WYNDOR GLASS CO. PROBLEM

The WYNDOR GLASS CO. produces high-quality glass products, including windows and glass doors. It has three plants. Aluminum frames and hardware are made in Plant 1, wood frames are made in Plant 2, and Plant 3 produces the glass and assembles the products. Because of declining earnings, top management has decided to revamp the company's product line. Unprofitable products are being discontinued, releasing production capacity to launch two new products having large sales potential:

Product 1: An 8-foot glass door with aluminum framing
Product 2: A 4×6 foot double-hung wood-framed window

WYNDOR GLASS CO. Example (Cont'd)

Product 1 requires some of the production capacity in Plants 1 and 3, but none in Plant 2. Product 2 needs only Plants 2 and 3. The marketing division has concluded that the company could sell as much of either product as could be produced by these plants. However, because both products would be competing for the same production capacity in Plant 3, it is not clear which *mix of the two products would be most profitable.*

Therefore, an OR team has been formed to determine what the *production rates should be for the two products in order to maximize their total profit, subject to the restrictions imposed by the limited production capacities*

Data for the Wyndor Glass Co. problem

Plant	Production Time per Batch, Hours — Product 1	Production Time per Batch, Hours — Product 2	Production Time Available per Week, Hours
1	1	0	4
2	0	2	12
3	3	2	18
Profit per batch	$3,000	$5,000	

Wyndor Glass Co. problem: Formulation

Maximize $Z = 3x_1 + 5x_2,$

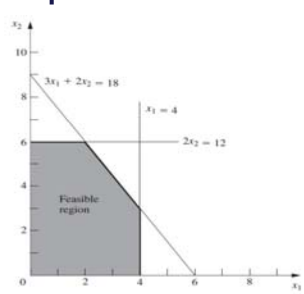
subject to the restrictions

$$x_1 \leq 4$$
$$2x_2 \leq 12$$
$$3x_1 + 2x_2 \leq 18$$

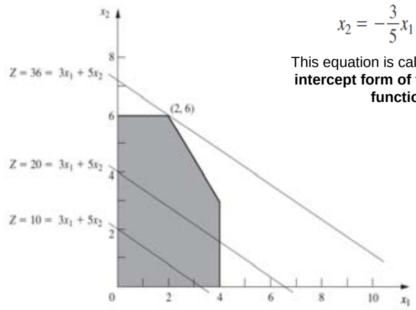
and

$$x_1 \geq 0, \quad x_2 \geq 0.$$

Wyndor Glass Co. problem: Graphical Solution



Wyndor Glass Co. problem: Graphical Solution



$$x_2 = -\frac{3}{5}x_1 + \frac{1}{5}Z$$

This equation is called the **slope-intercept form of the objective function**

LP Model: Common Terminology

Prototype Example	General Problem
Production capacities of plants 3 plants	Resources *m* resources
Production of products 2 products Production rate of product *j*, x_j	Activities *n* activities Level of activity *j*, x_j
Profit *Z*	Overall measure of performance *Z*

A Standard Form

$$\text{Maximize} \quad Z = c_1x_1 + c_2x_2 + \cdots + c_nx_n,$$

subject to the restrictions

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &\le b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &\le b_2 \\ &\vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &\le b_m, \end{aligned}$$

and

$$x_1 \ge 0, \quad x_2 \ge 0, \quad \ldots, \quad x_n \ge 0.$$

Data for LP

	Resource Usage per Unit of Activity				
	Activity				
Resource	1	2	. . .	*n*	Amount of Resource Available
1 2 . . . *m*	a_{11} a_{21} . . . a_{m1}	a_{12} a_{22} . . . a_{m2}	. . . .	a_{1n} a_{2n} . . . a_{mn}	b_1 b_2 . . . b_m
Contribution to *Z* per unit of activity	c_1	c_2	. . .	c_n	

Other Forms

1. Minimizing rather than maximizing the objective function
2. Some functional constraints with a greater-than-or-equal-to inequality
3. Some functional constraints in equation form
4. Deleting the nonnegativity constraints for some decision variables

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Solutions Terminology

- Any specification of values for the decision variables (x_1, x_2, . . . , x_n) is called a **solution, regardless** of whether it is a desirable or even an allowable choice.
- A feasible solution is a solution for which all the constraints are satisfied**.**
- An infeasible solution is a solution for which at least one constraint is violated.
- The feasible region is the collection of all feasible solutions.
- An optimal solution is a feasible solution that has the most favorable value of the objective function.

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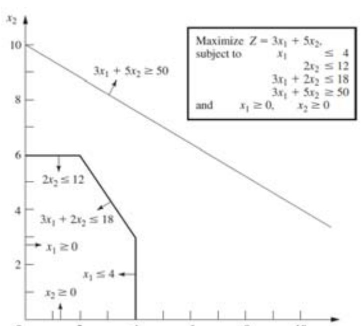
No Feasible Solutions



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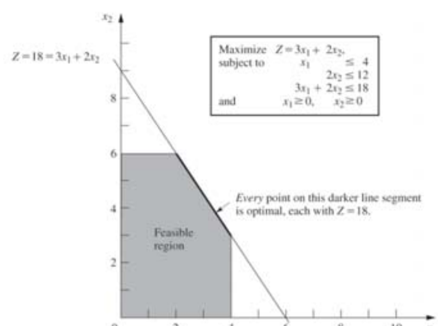
Multiple Optimal Solutions



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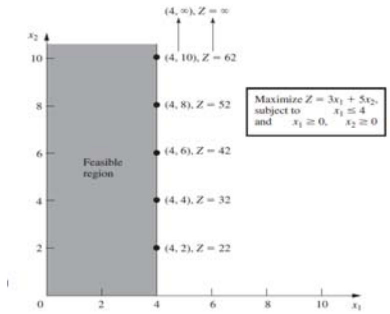
No Optimal Solutions



Solutions Terminology (Cont'd)

- A corner-point feasible (CPF) solution is a solution that lies at a corner of the feasible region.
- Relationship between optimal solutions and CPF solutions: For any linear programming problem with feasible solutions and a bounded feasible region:
  - the best CPF solution must be an optimal solution.
  - If the problem has multiple optimal solutions, at least two must be CPF solutions.

LP Assumptions

- **Proportionality assumption**: The contribution of each activity to the value of the objective function *Z* (and LHS of funtional constraint) is proportional to the level of the activity x_j.
- **Additivity assumption:** Every function in a linear programming model is the sum of the individual contributions of the respective activities.
- **Divisibility assumption:** Decision variables in a linear programming model are allowed to have any values, including noninteger values, that satisfy the functional and nonnegativity constraints.
- **Certainty assumption:** The value assigned to each parameter of a linear programming model is assumed to be a known constant.

Additional Example: Shader Electronics Problem

The Shader Electronics Company produces two products: (1) the Shader Walkman, a portable CD/DVD player, and (2) the Shader Watch-TV, a wristwatch-size internet-connected color television. The production process for each product is similar in that both require a certain number of hours of electronic work and a certain number of labor-hours in the assembly department. Each Walkman takes 4 hours of electronic work and 2 hours in the assembly shop. Each Watch-TV requires 3 hours in electronics and 1 hour in assembly. During the current production period, 240 hours of electronic time are available, and 100 hours of assembly department time are available. Each Walkman sold yields a profit of $7; each Watch-TV produced may be sold for a $5 profit. Shader's problem is to determine the best possible combination of Walkmans and Watch-TVs to manufacture to reach the maximum profit. This product-mix situation can be formulated as a linear programming problem.

Shader Electronics Example

Shader Electronics Company Problem Data

	Hours Required to Produce 1 Unit		
Department	Walkmans (X_1)	Watch-TVs (X_2)	Available Hours This Week
Electronic	4	3	240
Assembly	2	1	100
Profit per unit	$7	$5	

Shader Electronics Example

- Decision Variables:

  X_1 = number of Walkmans to be produced
  X_2 = number of Watch-TVs to be produced

- Objective function: Maximize profit = $\$7X_1 + \$5X_2$

- Constraints:

  First constraint: Electronic time used is ≤ Electronic time available.

  $4X_1 + 3X_2 \le 240$ (hours of electronic time)

  Second constraint: Assembly time used is ≤ Assembly time available.

  $2X_1 + 1X_2 \le 100$ (hours of assembly time)

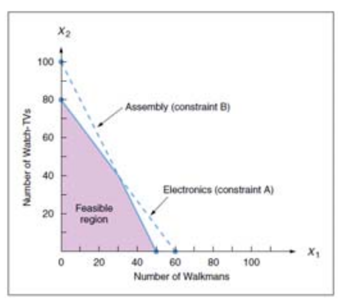
Shader Electronics Example

Maximize profit = $\$7X_1 + \$5X_2$

Subject to the constraints

$4X_1 + 3X_2 \le 240$ (*electronics constraint*)
$2X_1 + 1X_2 \le 100$ (*assembly constraint*)
$X_1 \ge 0$ (*number of Walkmans produced is greater than or equal to 0*)
$X_2 \ge 0$ (*number of Watch-TVs produced is greater than or equal to 0*)

Shader Electronics Example: Graphical Solution



Practice Problems

$$\begin{aligned} &Maximize \quad Z = 4x_1 + 3x_2 \\ &Subject\ to \\ &Material \quad 6x_1 + 4x_2 \le 48\ lb \\ &Labor \quad 4x_1 + 8x_2 \le 80\ hr \\ &x_1, x_2 \ge 0 \end{aligned}$$

$$\begin{aligned} &Maximize \quad Z = 6x_1 + 3x_2 \\ &Subject\ to \\ &Material \quad 20x_1 + 6x_2 \le 600\ lb \\ &Machinery \quad 25x_1 + 20x_2 \le 1000\ hr \\ &Labor \quad 20x_1 + 30x_2 \le 1200\ hr \\ &x_1, x_2 \ge 0 \end{aligned}$$

Additional Example: Design of Radiation Therapy

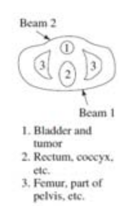
- MARY has just been diagnosed as having a cancer at a fairly advanced stage. Specifically, she has a large malignant tumor in the bladder area (a "whole bladder lesion"). Mary is to receive the most advanced medical care available to give her every possible chance for survival. This care will include extensive radiation therapy.
- Radiation therapy involves using an external beam treatment machine to pass ionizing radiation through the patient's body, damaging both cancerous and healthy tissues.
- Normally, several beams are precisely administered from different angles in a two dimensional plane. Due to attenuation, each beam delivers more radiation to the tissue near the entry point than to the tissue near the exit point. Scatter also causes some delivery of radiation to tissue outside the direct path of the beam.

Additional Example: Design of Radiation Therapy

- Because tumor cells are typically microscopically interspersed among healthy cells, the radiation dosage throughout the tumor region must be large enough to kill the malignant cells, which are slightly more radiosensitive, yet small enough to spare the healthy cells. At the same time, the aggregate dose to critical tissues must not exceed established tolerance levels, in order to prevent complications that can be more serious than the disease itself. For the same reason, the total dose to the entire healthy anatomy must be minimized.
- Because of the need to carefully balance all these factors, the design of radiation therapy is a very delicate process. The goal of the design is to select the combination of beams to be used, and the intensity of each one, to generate the best possible dose distribution. (The dose strength at any point in the body is measured in units called *kilorads.)*

Additional Example: Design of Radiation Therapy

- In Mary's case, the size and location of her tumor make the design of her treatment an even more delicate process than usual. Figure shows a diagram of a cross section of the tumor viewed from above, as well as nearby critical tissues to avoid. These tissues include critical organs (e.g., the rectum) as well as bony structures (e.g., the femurs and pelvis) that will attenuate the radiation.



Additional Example: Design of Radiation Therapy

TABLE 3.7 Data for the design of Mary's radiation therapy

Area	Fraction of Entry Dose Absorbed by Area (Average)		Restriction on Total Average Dosage, Kilorads
	Beam 1	Beam 2	
Healthy anatomy	0.4	0.5	Minimize
Critical tissues	0.3	0.1	≤ 2.7
Tumor region	0.5	0.5	= 6
Center of tumor	0.6	0.4	≥ 6

Additional Example: Design of Radiation Therapy

Minimize $Z = 0.4x_1 + 0.5x_2$,

subject to

$$0.3x_1 + 0.1x_2 \le 2.7$$
$$0.5x_1 + 0.5x_2 = 6$$
$$0.6x_1 + 0.4x_2 \ge 6$$

and

$$x_1 \ge 0, \quad x_2 \ge 0.$$