BANGLADESH ARMY INTERNATIONAL UNIVERSITY OF SCIENCE AND TECHNOLOGY (BAIUST), CUMILLA

Term Final Examination, Spring 2025

Department of Computer Science and Engineering

Level-1 Term-1

Course Code: MATH 111
Course Title: MATH I (Differential Calculus, Integral Calculus and Matrix

Exam Duration: 03 hours
Full Marks: 150

ANSWER SHEET

Examiner's Note: Answer any five (05) questions. Either three from Part-A and two from Part-B, or two from Part-A and three from Part-B.
Student's Note: All questions are answered below for reference and study purposes.

PART - A

Question 1(a)

Find the equation of the tangent at the point of the curve $x = a\cos^3\theta$, $y = b\sin^3\theta$.

Solution: To find the equation of the tangent, we first need the slope, $\frac{dy}{dx}$. We find this using the parametric chain rule: $\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta}$.

1. **Find $\frac{dx}{d\theta}$:**

$$\frac{dx}{d\theta} = a \cdot 3\cos^2\theta \cdot (-\sin\theta) = -3a\cos^2\theta\sin\theta$$

2. **Find $\frac{dy}{d\theta}$:**

$$\frac{dy}{d\theta} = b \cdot 3\sin^2\theta \cdot (\cos\theta) = 3b\sin^2\theta\cos\theta$$

3. **Find the slope $m = \frac{dy}{dx}$:**

$$m = \frac{3b\sin^2\theta\cos\theta}{-3a\cos^2\theta\sin\theta} = -\frac{b\sin\theta}{a\cos\theta}$$

4. **Use the point-slope formula for the tangent line:** $Y - y_1 = m(X - x_1)$, where $x_1 = a\cos^3\theta$ and $y_1 = b\sin^3\theta$.

$$Y - b\sin^3\theta = -\frac{b\sin\theta}{a\cos\theta}(X - a\cos^3\theta)$$

Multiply both sides by $a\cos\theta$:

$$a\cos\theta(Y - b\sin^3\theta) = -b\sin\theta(X - a\cos^3\theta)$$

$$aY\cos\theta - ab\sin^3\theta\cos\theta = -bX\sin\theta + ab\sin\theta\cos^3\theta$$

Rearrange the terms:

$$bX\sin\theta + aY\cos\theta = ab\sin\theta\cos^3\theta + ab\sin^3\theta\cos\theta$$

Factor out $ab\sin\theta\cos\theta$ on the right side:

$$bX\sin\theta + aY\cos\theta = ab\sin\theta\cos\theta(\cos^2\theta + \sin^2\theta)$$

Since $\cos^2\theta + \sin^2\theta = 1$:

$$bX\sin\theta + aY\cos\theta = ab\sin\theta\cos\theta$$

Divide by $ab\sin\theta\cos\theta$ (assuming θ is not $0, \pi/2$, etc.):

$$\frac{bX\sin\theta}{ab\sin\theta\cos\theta} + \frac{aY\cos\theta}{ab\sin\theta\cos\theta} = 1$$

$$\frac{X}{a\cos\theta} + \frac{Y}{b\sin\theta} = 1$$

Answer: The equation of the tangent is $\frac{X}{a\cos\theta} + \frac{Y}{b\sin\theta} = 1$.

Question 1(b)

If $u = \sin^{-1}\frac{x}{y} + \tan^{-1}\frac{y}{x}$**, show that** $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 0$**.**

Solution: We will find the partial derivatives $\frac{\partial u}{\partial x}$ and $\frac{\partial u}{\partial y}$ and then compute the required sum.

1. **Find** $\frac{\partial u}{\partial x}$**:**

$$\frac{\partial u}{\partial x} = \frac{1}{\sqrt{1-(x/y)^2}} \cdot \left(\frac{1}{y}\right) + \frac{1}{1+(y/x)^2} \cdot \left(-\frac{y}{x^2}\right)$$

$$= \frac{1}{\sqrt{\frac{y^2-x^2}{y^2}}} \cdot \left(\frac{1}{y}\right) - \frac{1}{\frac{x^2+y^2}{x^2}} \cdot \left(\frac{y}{x^2}\right)$$

$$= \frac{y}{\sqrt{y^2-x^2}} \cdot \left(\frac{1}{y}\right) - \frac{x^2}{x^2+y^2} \cdot \left(\frac{y}{x^2}\right)$$

$$\frac{\partial u}{\partial x} = \frac{1}{\sqrt{y^2-x^2}} - \frac{y}{x^2+y^2}$$

Therefore, $x\frac{\partial u}{\partial x} = \frac{x}{\sqrt{y^2-x^2}} - \frac{xy}{x^2+y^2}$.

2. **Find $\frac{\partial u}{\partial y}$:**

$$\frac{\partial u}{\partial y} = \frac{1}{\sqrt{1-(x/y)^2}} \cdot \left(-\frac{x}{y^2}\right) + \frac{1}{1+(y/x)^2} \cdot \left(\frac{1}{x}\right)$$

$$= \frac{y}{\sqrt{y^2-x^2}} \cdot \left(-\frac{x}{y^2}\right) + \frac{x^2}{x^2+y^2} \cdot \left(\frac{1}{x}\right)$$

$$\frac{\partial u}{\partial y} = -\frac{x}{y\sqrt{y^2-x^2}} + \frac{x}{x^2+y^2}$$

Therefore, $y\frac{\partial u}{\partial y} = -\frac{x}{\sqrt{y^2-x^2}} + \frac{xy}{x^2+y^2}$.

3. **Sum the terms:**

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \left(\frac{x}{\sqrt{y^2-x^2}} - \frac{xy}{x^2+y^2}\right) + \left(-\frac{x}{\sqrt{y^2-x^2}} + \frac{xy}{x^2+y^2}\right)$$

All terms cancel out.

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 0$$

(**Note:** This is also a homogeneous function of degree $n = 0$. By Euler's Theorem, $xu_x + yu_y = nu = 0 \cdot u = 0$, which confirms our result.)

Question 2(a)

Compute the integral $\int \frac{\cos x}{\sin^2 x + 4\sin x + 3} dx$.

Solution: 1. **Substitution:** Let $u = \sin x$. Then $du = \cos x dx$. The integral becomes:

$$\int \frac{du}{u^2+4u+3}$$

2. **Partial Fractions:** Factor the denominator: $u^2+4u+3 = (u+1)(u+3)$. We set up the partial fraction decomposition:

$$\frac{1}{(u+1)(u+3)} = \frac{A}{u+1} + \frac{B}{u+3}$$

$$1 = A(u+3) + B(u+1)$$

- Let $u = -1$: $1 = A(-1+3) = 2A \implies A = \frac{1}{2}$
- Let $u = -3$: $1 = B(-3+1) = -2B \implies B = -\frac{1}{2}$

3. **Integrate:**

$$\int \left(\frac{1/2}{u+1} - \frac{1/2}{u+3}\right) du = \frac{1}{2}\int \frac{du}{u+1} - \frac{1}{2}\int \frac{du}{u+3}$$

$$= \frac{1}{2}\ln|u+1| - \frac{1}{2}\ln|u+3| + C$$

$$= \frac{1}{2}(\ln|u+1| - \ln|u+3|) + C = \frac{1}{2}\ln\left|\frac{u+1}{u+3}\right| + C$$

4. **Substitute back:** Replace u with $\sin x$.

$$\frac{1}{2}\ln\left|\frac{\sin x+1}{\sin x+3}\right| + C$$

Answer: $\frac{1}{2}\ln\left|\frac{\sin x+1}{\sin x+3}\right| + C$

Question 2(b)

Explain the integral $\int_0^{\pi/4} x \cos x \cos 3x dx$.

Solution: 1. **Product-to-Sum:** First, simplify $\cos x \cos 3x$ using the identity $\cos A \cos B = \frac{1}{2}[\cos(A+B) + \cos(A-B)]$.

$$\cos x \cos 3x = \frac{1}{2}[\cos(x+3x) + \cos(x-3x)] = \frac{1}{2}[\cos(4x) + \cos(-2x)]$$

Since $\cos(-2x) = \cos(2x)$, this simplifies to $\frac{1}{2}(\cos 4x + \cos 2x)$. 2. **Rewrite Integral:**

$$I = \int_0^{\pi/4} x \cdot \frac{1}{2}(\cos 4x + \cos 2x)dx = \frac{1}{2}\int_0^{\pi/4} x(\cos 4x + \cos 2x)dx$$

3. **Integration by Parts:** Use $\int udv = uv - \int vdu$.

- Let $u = x \implies du = dx$
- Let $dv = (\cos 4x + \cos 2x)dx \implies v = \int(\cos 4x + \cos 2x)dx = \frac{\sin 4x}{4} + \frac{\sin 2x}{2}$

4. **Apply Parts:**

$$I = \frac{1}{2}\left[\left(x\left(\frac{\sin 4x}{4} + \frac{\sin 2x}{2}\right)\right)\Bigg|_0^{\pi/4} - \int_0^{\pi/4}\left(\frac{\sin 4x}{4} + \frac{\sin 2x}{2}\right)dx\right]$$

5. **Evaluate Terms:**

- **First Term (uv):**

$$\text{At } x = \pi/4: \quad \frac{\pi}{4}\left(\frac{\sin(\pi)}{4} + \frac{\sin(\pi/2)}{2}\right) = \frac{\pi}{4}\left(0 + \frac{1}{2}\right) = \frac{\pi}{8}$$

$$\text{At } x = 0: \quad 0 \cdot (\dots) = 0$$

Value of first term = $\frac{\pi}{8} - 0 = \frac{\pi}{8}$.

- **Second Term ($\int vdu$):**

$$\int_0^{\pi/4}\left(\frac{\sin 4x}{4} + \frac{\sin 2x}{2}\right)dx = \left[-\frac{\cos 4x}{16} - \frac{\cos 2x}{4}\right]\Bigg|_0^{\pi/4}$$

$$\text{At } x = \pi/4: \quad \left(-\frac{\cos(\pi)}{16} - \frac{\cos(\pi/2)}{4}\right) = \left(-\frac{-1}{16} - 0\right) = \frac{1}{16}$$

$$\text{At } x = 0: \quad \left(-\frac{\cos(0)}{16} - \frac{\cos(0)}{4}\right) = \left(-\frac{1}{16} - \frac{1}{4}\right) = -\frac{1+4}{16} = -\frac{5}{16}$$

Value of second term = $\frac{1}{16} - (-\frac{5}{16}) = \frac{6}{16} = \frac{3}{8}$.

6. **Final Calculation:**

$$I = \frac{1}{2}\left[(\text{First Term}) - (\text{Second Term})\right] = \frac{1}{2}\left[\frac{\pi}{8} - \frac{3}{8}\right] = \frac{\pi - 3}{16}$$

Answer: $\frac{\pi-3}{16}$

Question 2(c)

Indicate the solution of the integral $\int_0^{\pi/2} \frac{dx}{1+\cos x}$.

Solution: 1. **Half-Angle Identity:** Use the identity $1 + \cos x = 2\cos^2(x/2)$.

$$I = \int_0^{\pi/2} \frac{dx}{2\cos^2(x/2)} = \frac{1}{2}\int_0^{\pi/2} \sec^2(x/2)dx$$

2. **Integrate:** The integral of $\sec^2(au)$ is $\frac{1}{a}\tan(au)$.

$$I = \frac{1}{2}\left[\frac{\tan(x/2)}{1/2}\right]\Bigg|_0^{\pi/2} = \frac{1}{2}\left[2\tan(x/2)\right]\Bigg|_0^{\pi/2} = \left[\tan(x/2)\right]\Bigg|_0^{\pi/2}$$

3. **Evaluate Limits:**

$$I = \tan\left(\frac{\pi/2}{2}\right) - \tan\left(\frac{0}{2}\right) = \tan(\pi/4) - \tan(0)$$

$$I = 1 - 0 = 1$$

Answer: 1

Question 3(a)

Demonstrate the area of the quadrant of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ between the major and minor axes.

Solution: The area of the first quadrant is found by integrating the function $y = f(x)$ from $x = 0$ to $x = a$.

1. **Solve for y:**

$$\frac{y^2}{b^2} = 1 - \frac{x^2}{a^2} = \frac{a^2 - x^2}{a^2}$$

$$y^2 = \frac{b^2}{a^2}(a^2 - x^2) \implies y = \frac{b}{a}\sqrt{a^2 - x^2} \quad \text{(for the first quadrant)}$$

2. **Set up the Area Integral:**

$$A = \int_0^a y dx = \int_0^a \frac{b}{a}\sqrt{a^2 - x^2} dx = \frac{b}{a}\int_0^a \sqrt{a^2 - x^2} dx$$

3. **Trigonometric Substitution:** Let $x = a \sin\theta$. Then $dx = a\cos\theta d\theta$.
    - When $x = 0$, $0 = a\sin\theta \implies \theta = 0$.
    - When $x = a$, $a = a\sin\theta \implies \theta = \pi/2$.

The integral becomes:

$$A = \frac{b}{a}\int_0^{\pi/2} \sqrt{a^2 - a^2\sin^2\theta} \cdot (a\cos\theta d\theta)$$

$$A = \frac{b}{a}\int_0^{\pi/2} \sqrt{a^2\cos^2\theta} \cdot (a\cos\theta d\theta) = \frac{b}{a}\int_0^{\pi/2} (a\cos\theta)(a\cos\theta d\theta)$$

$$A = \frac{b}{a}\int_0^{\pi/2} a^2\cos^2\theta d\theta = ab\int_0^{\pi/2} \cos^2\theta d\theta$$

4. **Integrate $\cos^2\theta$:** Use the identity $\cos^2\theta = \frac{1+\cos 2\theta}{2}$.

$$A = ab\int_0^{\pi/2} \left(\frac{1}{2} + \frac{\cos 2\theta}{2}\right) d\theta = \frac{ab}{2}\left[\theta + \frac{\sin 2\theta}{2}\right]\Bigg|_0^{\pi/2}$$

5. **Evaluate Limits:**

$$A = \frac{ab}{2}\left[\left(\frac{\pi}{2} + \frac{\sin(\pi)}{2}\right) - \left(0 + \frac{\sin(0)}{2}\right)\right]$$

$$A = \frac{ab}{2}\left[\left(\frac{\pi}{2} + 0\right) - (0 + 0)\right] = \frac{ab}{2}\left(\frac{\pi}{2}\right) = \frac{\pi ab}{4}$$

Answer: The area of the quadrant is $\frac{\pi ab}{4}$.

Question 3(b)

Determine the length of the arc of the cycloid $x = a(\theta + sin\ \theta)$, $y = a(1 - cos\ \theta)$ measured from the origin.

Solution: The arc length L of a parametric curve is $L = \int \sqrt{(\frac{dx}{d\theta})^2 + (\frac{dy}{d\theta})^2} d\theta$.

1. **Find Derivatives:**

$$\frac{dx}{d\theta} = a(1 + \cos\theta)$$

$$\frac{dy}{d\theta} = a(0 - (-\sin\theta)) = a\sin\theta$$

2. **Sum of Squares:**

$$\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2 = a^2(1 + \cos\theta)^2 + (a\sin\theta)^2$$

$$= a^2(1 + 2\cos\theta + \cos^2\theta) + a^2\sin^2\theta$$
$$= a^2(1 + 2\cos\theta + \cos^2\theta + \sin^2\theta)$$

Since $\cos^2\theta + \sin^2\theta = 1$:

$$= a^2(1 + 2\cos\theta + 1) = a^2(2 + 2\cos\theta) = 2a^2(1 + \cos\theta)$$

3. **Half-Angle Identity:** Use $1 + \cos\theta = 2\cos^2(\theta/2)$.

$$2a^2(1 + \cos\theta) = 2a^2(2\cos^2(\theta/2)) = 4a^2\cos^2(\theta/2)$$

4. **Integrate:** The origin corresponds to $\theta = 0$. We integrate from 0 to a general θ.

$$L = \int_0^\theta \sqrt{4a^2\cos^2(\theta/2)}d\theta = \int_0^\theta 2a\cos(\theta/2)d\theta$$

(We take the positive root as θ is in the first part of the arc).

$$L = 2a\left[\frac{\sin(\theta/2)}{1/2}\right]\Bigg|_0^\theta = 2a\left[2\sin(\theta/2)\right]\Big|_0^\theta$$
$$L = 4a\sin(\theta/2) - 4a\sin(0) = 4a\sin(\theta/2)$$

Answer: The arc length from the origin is $L = 4a\sin(\theta/2)$.

PART - B

Question 1(a)

Show that $\int_0^{\pi/2} \sin^4\theta \cos^6\theta d\theta = \frac{3\pi}{512}$.

Solution: We use Wallis's Formula for integrals of the form $\int_0^{\pi/2} \sin^m x \cos^n x dx$. Here, $m = 4$ and $n = 6$. Since both m and n are even, the formula is:

$$I = \frac{(m-1)!!(n-1)!!}{(m+n)!!} \cdot \frac{\pi}{2}$$

Where !! denotes the double factorial.

- $m - 1 = 3 \implies (m-1)!! = 3 \cdot 1 = 3$
- $n - 1 = 5 \implies (n-1)!! = 5 \cdot 3 \cdot 1 = 15$
- $m + n = 10 \implies (m+n)!! = 10 \cdot 8 \cdot 6 \cdot 4 \cdot 2 = 3840$

$$I = \frac{(3)(15)}{3840} \cdot \frac{\pi}{2} = \frac{45}{3840} \cdot \frac{\pi}{2}$$

Simplify the fraction $\frac{45}{3840}$:

$$\frac{45}{3840} = \frac{9}{768} \text{ (div by 5)} = \frac{3}{256} \text{ (div by 3)}$$

Now, substitute this back:

$$I = \frac{3}{256} \cdot \frac{\pi}{2} = \frac{3\pi}{512}$$

This matches the required result. **(Proven)**

Question 1(b)

Estimate the solution of the integral $\int \log(1+x)^{(1+x)} dx$.

Solution: (Note: The question as written, $\log(1+x)^{(1+x)}$, appears to be a typo. We will solve the likely intended integral: $\int (1+x)\ln(1+x)dx$. We assume log means natural log, ln.)

1. **Substitution:** Let $u = 1 + x$. Then $du = dx$. The integral becomes:

$$\int u \ln(u) du$$

2. **Integration by Parts:** Use $\int fg' = fg - \int f'g$.

- Let $f = \ln u \implies f' = \frac{1}{u}$
- Let $g' = u \implies g = \frac{u^2}{2}$

$$I = (\ln u)\left(\frac{u^2}{2}\right) - \int \left(\frac{1}{u}\right)\left(\frac{u^2}{2}\right) du$$

$$I = \frac{u^2 \ln u}{2} - \int \frac{u}{2} du$$

$$I = \frac{u^2 \ln u}{2} - \frac{1}{2}\left(\frac{u^2}{2}\right) + C = \frac{u^2 \ln u}{2} - \frac{u^2}{4} + C$$

$$I = \frac{u^2}{4}(2\ln u - 1) + C$$

3. **Substitute back:** Replace u with $(1 + x)$.

$$I = \frac{(1+x)^2}{4}(2\ln(1+x) - 1) + C$$

Answer: $\frac{(1+x)^2}{4}(2\ln(1+x) - 1) + C$

Question 1(c)

Find the solution of the integral $\int \frac{\cos^2 x}{\sin^4 x} dx$.

Solution: 1. **Rewrite in terms of** $\cot x$ **and** $\csc x$**:**

$$I = \int \frac{\cos^2 x}{\sin^2 x} \cdot \frac{1}{\sin^2 x} dx = \int \cot^2 x \csc^2 x dx$$

2. **Substitution:** Let $u = \cot x$. Then $du = -\csc^2 x dx \implies -du = \csc^2 x dx$. 3. **Integrate:**

$$I = \int u^2(-du) = -\int u^2 du$$

$$I = -\frac{u^3}{3} + C$$

4. **Substitute back:** Replace u with $\cot x$.

$$I = -\frac{\cot^3 x}{3} + C$$

Answer: $-\frac{1}{3}\cot^3 x + C$

Question 2(a)

Prove that the matrix $A = \begin{bmatrix} 2 & -2 & -4 \\ -1 & 3 & 4 \\ 1 & -2 & -3 \end{bmatrix}$ is idempotent.

Solution: A matrix A is idempotent if $A^2 = A$. We will compute A^2.

$$A^2 = A \cdot A = \begin{bmatrix} 2 & -2 & -4 \\ -1 & 3 & 4 \\ 1 & -2 & -3 \end{bmatrix} \begin{bmatrix} 2 & -2 & -4 \\ -1 & 3 & 4 \\ 1 & -2 & -3 \end{bmatrix}$$

$$C_{11} = (2)(2) + (-2)(-1) + (-4)(1) = 4 + 2 - 4 = 2$$
$$C_{12} = (2)(-2) + (-2)(3) + (-4)(-2) = -4 - 6 + 8 = -2$$
$$C_{13} = (2)(-4) + (-2)(4) + (-4)(-3) = -8 - 8 + 12 = -4$$

$$C_{21} = (-1)(2) + (3)(-1) + (4)(1) = -2 - 3 + 4 = -1$$
$$C_{22} = (-1)(-2) + (3)(3) + (4)(-2) = 2 + 9 - 8 = 3$$
$$C_{23} = (-1)(-4) + (3)(4) + (4)(-3) = 4 + 12 - 12 = 4$$

$$C_{31} = (1)(2) + (-2)(-1) + (-3)(1) = 2 + 2 - 3 = 1$$
$$C_{32} = (1)(-2) + (-2)(3) + (-3)(-2) = -2 - 6 + 6 = -2$$
$$C_{33} = (1)(-4) + (-2)(4) + (-3)(-3) = -4 - 8 + 9 = -3$$

So, $A^2 = \begin{bmatrix} 2 & -2 & -4 \\ -1 & 3 & 4 \\ 1 & -2 & -3 \end{bmatrix}$. Since $A^2 = A$, the matrix A is idempotent. **(Proven)**

Question 2(b)

Solve the system of linear equations with the help of matrices: $x + y + z = 6$ $x - y + z = 2$ $x + y - z = 1$

Solution: We write the system as $AX = B$, where $X = A^{-1}B$.

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{bmatrix}, \quad X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}, \quad B = \begin{bmatrix} 6 \\ 2 \\ 1 \end{bmatrix}$$

1. **Find** det(A)**:**

$$\text{det}(A) = 1((-1)(-1) - (1)(1)) - 1((1)(-1) - (1)(1)) + 1((1)(1) - (-1)(1))$$
$$= 1(1 - 1) - 1(-1 - 1) + 1(1 + 1) = 0 - (-2) + 2 = 4$$

2. **Find Adjoint of A:** This is the transpose of the cofactor matrix.

$$C_{11} = 0, \quad C_{12} = -(-2) = 2, \quad C_{13} = 2$$
$$C_{21} = -(-2) = 2, \quad C_{22} = -2, \quad C_{23} = -(0) = 0$$
$$C_{31} = 2, \quad C_{32} = -(0) = 0, \quad C_{33} = -2$$

Cofactor Matrix $C = \begin{bmatrix} 0 & 2 & 2 \\ 2 & -2 & 0 \\ 2 & 0 & -2 \end{bmatrix}$. Adjoint Matrix adj$(A) = C^T = \begin{bmatrix} 0 & 2 & 2 \\ 2 & -2 & 0 \\ 2 & 0 & -2 \end{bmatrix}$. 3. **Find Inverse** A^{-1}**:**

$$A^{-1} = \frac{1}{\text{det}(A)}\text{adj}(A) = \frac{1}{4}\begin{bmatrix} 0 & 2 & 2 \\ 2 & -2 & 0 \\ 2 & 0 & -2 \end{bmatrix}$$

4. **Solve for X:**

$$X = A^{-1}B = \frac{1}{4}\begin{bmatrix} 0 & 2 & 2 \\ 2 & -2 & 0 \\ 2 & 0 & -2 \end{bmatrix} \begin{bmatrix} 6 \\ 2 \\ 1 \end{bmatrix}$$

$$X = \frac{1}{4}\begin{bmatrix} (0)(6)+(2)(2)+(2)(1) \\ (2)(6)+(-2)(2)+(0)(1) \\ (2)(6)+(0)(2)+(-2)(1) \end{bmatrix} = \frac{1}{4}\begin{bmatrix} 0+4+2 \\ 12-4+0 \\ 12+0-2 \end{bmatrix}$$

$$X = \frac{1}{4}\begin{bmatrix} 6 \\ 8 \\ 10 \end{bmatrix} = \begin{bmatrix} 6/4 \\ 8/4 \\ 10/4 \end{bmatrix} = \begin{bmatrix} 1.5 \\ 2 \\ 2.5 \end{bmatrix}$$

Answer: $x = 1.5$, $y = 2$, $z = 2.5$.

Question 3(a)

Determine the row echelon form for the system of linear equations: $x_1 + 2x_2 - x_3 = 2$ $2x_1 + x_2 + x_3 = 1$ $x_1 + 5x_2 - 4x_3 = 5$

Solution: We form the augmented matrix $[A|B]$ and perform row operations.

$$[A|B] = \begin{bmatrix} 1 & 2 & -1 & | & 2 \\ 2 & 1 & 1 & | & 1 \\ 1 & 5 & -4 & | & 5 \end{bmatrix}$$

$R_2 \to R_2 - 2R_1$ $R_3 \to R_3 - R_1$

$$\sim \begin{bmatrix} 1 & 2 & -1 & | & 2 \\ 0 & 1-4 & 1-(-2) & | & 1-4 \\ 0 & 5-2 & -4-(-1) & | & 5-2 \end{bmatrix} = \begin{bmatrix} 1 & 2 & -1 & | & 2 \\ 0 & -3 & 3 & | & -3 \\ 0 & 3 & -3 & | & 3 \end{bmatrix}$$

$R_2 \to R_2/(-3)$

$$\sim \begin{bmatrix} 1 & 2 & -1 & | & 2 \\ 0 & 1 & -1 & | & 1 \\ 0 & 3 & -3 & | & 3 \end{bmatrix}$$

$R_3 \to R_3 - 3R_2$

$$\sim \begin{bmatrix} 1 & 2 & -1 & | & 2 \\ 0 & 1 & -1 & | & 1 \\ 0 & 0 & 0 & | & 0 \end{bmatrix}$$

This is a row echelon form. (To get reduced row echelon form, we would do $R_1 \to R_1 - 2R_2$, which gives $\begin{bmatrix} 1 & 0 & 1 & | & 0 \\ \dots \end{bmatrix}$).

Answer: The row echelon form is $\begin{bmatrix} 1 & 2 & -1 & | & 2 \\ 0 & 1 & -1 & | & 1 \\ 0 & 0 & 0 & | & 0 \end{bmatrix}$.

Question 3(b)

Calculate the eigenvalues of the matrix $A = \begin{bmatrix} 1 & 2 & -1 \\ 0 & -2 & 0 \\ 0 & -5 & 2 \end{bmatrix}$**.**

Solution: The eigenvalues (λ) of a matrix are the roots of the characteristic equation $\det(A - \lambda I) = 0$. For a triangular matrix (upper or lower), the eigenvalues are simply the entries on the main diagonal. The given matrix A is a lower triangular matrix. Therefore, the eigenvalues are the diagonal entries: $1, -2, 2$.

Answer: The eigenvalues are $\lambda_1 = 1$, $\lambda_2 = -2$, $\lambda_3 = 2$.

Question 3(c)

If $A = \begin{bmatrix} 1 & 1+i \\ 2-3i & i \end{bmatrix}$ **and** $B = \begin{bmatrix} 2-i & i \\ 1+5i & 3 \end{bmatrix}$**, then prove that** $\overline{A+B} = \overline{A} + \overline{B}$**.**

Solution: We will compute the Left Hand Side (LHS) and Right Hand Side (RHS) separately.

1. **LHS:** $\overline{A+B}$: First, find $A+B$:

$$A + B = \begin{bmatrix} 1 & 1+i \\ 2-3i & i \end{bmatrix} + \begin{bmatrix} 2-i & i \\ 1+5i & 3 \end{bmatrix}$$

$$A + B = \begin{bmatrix} 1+(2-i) & (1+i)+i \\ (2-3i)+(1+5i) & i+3 \end{bmatrix} = \begin{bmatrix} 3-i & 1+2i \\ 3+2i & 3+i \end{bmatrix}$$

Now, find the conjugate of the result:

$$\overline{A+B} = \begin{bmatrix} \overline{3-i} & \overline{1+2i} \\ \overline{3+2i} & \overline{3+i} \end{bmatrix} = \begin{bmatrix} 3+i & 1-2i \\ 3-2i & 3-i \end{bmatrix}$$

2. **RHS:** $\overline{A} + \overline{B}$: First, find the conjugates $\overline{A}$ and $\overline{B}$:

$$\overline{A} = \begin{bmatrix} \overline{1} & \overline{1+i} \\ \overline{2-3i} & \overline{i} \end{bmatrix} = \begin{bmatrix} 1 & 1-i \\ 2+3i & -i \end{bmatrix}$$

$$\overline{B} = \begin{bmatrix} \overline{2-i} & \overline{i} \\ \overline{1+5i} & \overline{3} \end{bmatrix} = \begin{bmatrix} 2+i & -i \\ 1-5i & 3 \end{bmatrix}$$

Now, add the conjugates:

$$\overline{A} + \overline{B} = \begin{bmatrix} 1 & 1-i \\ 2+3i & -i \end{bmatrix} + \begin{bmatrix} 2+i & -i \\ 1-5i & 3 \end{bmatrix}$$

$$\overline{A} + \overline{B} = \begin{bmatrix} 1+(2+i) & (1-i)+(-i) \\ (2+3i)+(1-5i) & (-i)+3 \end{bmatrix} = \begin{bmatrix} 3+i & 1-2i \\ 3-2i & 3-i \end{bmatrix}$$

3. **Conclusion:** Since LHS = RHS, we have proven that $\overline{A+B} = \overline{A} + \overline{B}$. **(Proven)**