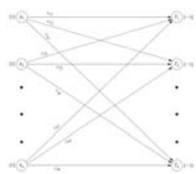
Lecture 12: Transportation & Assignment Problems



Outline

- Transportation Problems Review
- Additional Examples of Transportation Problem
- Assignment Problem

General Description of a Transportation Problem

1. A set of m supply points from which a good is shipped. Supply point i can supply at most s_i units.
2. Set of n demand points to which the good is shipped. Demand point j must receive at least d_j units of the shipped good.
3. Each unit produced at supply point i and shipped to demand point j incurs a variable cost of c_{ij}.

General Formulation

$$\min \sum_{i=1}^{i=m} \sum_{j=1}^{j=n} c_{ij} x_{ij}$$

$$\text{s.t.} \quad \sum_{j=1}^{j=n} x_{ij} \le s_i \quad (i = 1, 2, \ldots, m) \quad \text{(Supply constraints)}$$

$$\sum_{i=1}^{i=m} x_{ij} \ge d_j \quad (j = 1, 2, \ldots, n) \quad \text{(Demand constraints)}$$

$$x_{ij} \ge 0 \quad (i = 1, 2, \ldots, m; j = 1, 2, \ldots, n)$$

Balanced Transportation Problem

- Total supply equals total demand

$$\sum_{i=1}^{i=m} s_i = \sum_{j=1}^{j=n} d_j \quad \text{(Feasible Solutions Property)}$$

$$\min \sum_{i=1}^{i=m} \sum_{j=1}^{j=n} c_{ij} x_{ij}$$

$$\text{s.t.} \quad \sum_{j=1}^{j=n} x_{ij} = s_i \quad (i = 1, 2, \ldots, m) \quad \text{(Supply constraints)}$$

$$\sum_{i=1}^{i=m} x_{ij} = d_j \quad (j = 1, 2, \ldots, n) \quad \text{(Demand constraints)}$$

$$x_{ij} \geq 0 \quad (i = 1, 2, \ldots, m;\ j = 1, 2, \ldots, n)$$

A Transportation Tableau

					Supply
	c_{11}	c_{12}	$\cdots$	c_{1n}	s_1
	c_{21}	c_{22}	$\cdots$	c_{2n}	s_2
	$\vdots$	$\vdots$		$\vdots$	$\vdots$
	c_{m1}	c_{m2}	$\cdots$	c_{mn}	s_m
Demand	d_1	d_2	$\cdots$	d_n	

Transportation Problem Assumptions

- **The requirements assumption:** Each source has a fixed *supply of units,* where this entire supply must be distributed to the destinations. Similarly, each destination has a fixed *demand for units,* where this entire demand must be received from the sources.
- **The feasible solutions property:** A transportation problem will have feasible solutions if and only if $\sum_{i=1}^{m} s_i = \sum_{j=1}^{n} d_j$
- **The cost assumption:** The cost of distributing units from any particular source to any particular destination is *directly proportional to the number of units distributed.* Therefore, this cost is just the *unit cost of distribution times the number of units distributed.*

Integer solutions property

- For transportation problems where every s_i *and* d_j have an integer value, all the basic variables (allocations) in *every basic feasible* (BF) solution (including an optimal one) also have *integer values*.

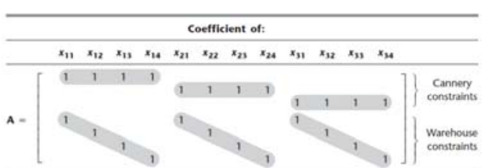
Transportation Problems: Example 1

- One of the main products of the P & T COMPANY is canned peas. The peas are prepared at three canneries (near Bellingham, Washington; Eugene, Oregon; and Albert Lea, Minnesota) and then shipped by truck to four distributing warehouses in the western United States (Sacramento, California; Salt Lake City, Utah; Rapid City, South Dakota; and Albuquerque, New Mexico). Because the shipping costs are a major expense, management is initiating a study to reduce them as much as possible. For the upcoming season, an estimate has been made of the output from each cannery, and each warehouse has been allocated a certain amount from the total supply of peas. This information (in units of truckloads), along with the shipping cost per truckload for each cannery-warehouse combination, is given in the following table. Thus, there are a total of 300 truckloads to be shipped. The problem now is to determine which plan for assigning these shipments to the various cannery-warehouse combinations would *minimize the total shipping cost*.

Shipping data for P & T Co.

		Shipping Cost ($) per Truckload				
		Warehouse				
		1	2	3	4	Output
Cannery	1	464	513	654	867	75
	2	352	416	690	791	125
	3	995	682	388	685	100
Allocation		80	65	70	85	

Constraint coefficients for P & T Co.



Formulation for P & T Co.

Minimize $Z = 464x_{11} + 513x_{12} + 654x_{13} + 867x_{14} + 352x_{21} + 416x_{22} + 690x_{23} + 791x_{24} + 995x_{31} + 682x_{32} + 388x_{33} + 685x_{34}$,

subject to the constraints

$$
\begin{aligned}
x_{11} + x_{12} + x_{13} + x_{14} &= 75 \\
x_{21} + x_{22} + x_{23} + x_{24} &= 125 \\
x_{31} + x_{32} + x_{33} + x_{34} &= 100 \\
x_{11} + x_{21} + x_{31} &= 80 \\
x_{12} + x_{22} + x_{32} &= 65 \\
x_{13} + x_{23} + x_{33} &= 70 \\
x_{14} + x_{24} + x_{34} &= 85
\end{aligned}
$$

and

$x_{ij} \geq 0 \quad (i = 1, 2, 3; j = 1, 2, 3, 4).$

Transportation Problems: Example 2

- The NORTHERN AIRPLANE COMPANY builds commercial airplanes for various airline companies around the world. The last stage in the production process is to produce the jet engines and then to install them (a very fast operation) in the completed airplane frame. The company has been working under some contracts to deliver a considerable number of airplanes in the near future, and the production of the jet engines for these planes must now be scheduled for the next 4 months.
- To meet the contracted dates for delivery, the company must supply engines for installation in the quantities indicated in the second column of following table. Thus, the cumulative number of engines produced by the end of months 1, 2, 3, and 4 must be at least 10, 25, 50, and 70, respectively.
- The facilities that will be available for producing the engines vary according to other production, maintenance, and renovation work scheduled during this period. The resulting monthly differences in the maximum number that can be produced and the cost (in millions of dollars) of producing each one are given in the third and fourth columns of the following table.

Production scheduling data for Northern Airplane Co.

Month	Scheduled Installations	Maximum Production	Unit Cost* of Production	Unit Cost* of Storage
1	10	25	1.08	0.015
2	15	35	1.11	0.015
3	25	30	1.10	0.015
4	20	10	1.13	

*Cost is expressed in millions of dollars.

Because of the variations in production costs, it may well be worthwhile to produce some of the engines a month or more before they are scheduled for installation, and this possibility is being considered. The drawback is that such engines must be stored until the scheduled installation (the airplane frames will not be ready early) at a storage cost of $15,000 per month (including interest on expended capital) for each engine, as shown in the rightmost column of the table above.

The production manager wants a schedule developed for the number of engines to be produced in each of the 4 months so that the total of the production and storage costs will be minimized.

Incomplete parameter table for Northern Airplane Co.

		Cost per Unit Distributed				
		Destination				
		1	2	3	4	Supply
Source	1	1.080	1.095	1.110	1.125	?
	2	?	1.110	1.125	1.140	?
	3	?	?	1.100	1.115	?
	4	?	?	?	1.130	?
Demand		10	15	25	20	

Complete parameter table for Northern Airplane Co.

		Cost per Unit Distributed					
		Destination					
		1	2	3	4	5(*D*)	Supply
Source	1	1.080	1.095	1.110	1.125	0	25
	2	*M*	1.110	1.125	1.140	0	35
	3	*M*	*M*	1.100	1.115	0	30
	4	*M*	*M*	*M*	1.130	0	10
Demand		10	15	25	20	30	

Transportation Problems: Example 3

- METRO WATER DISTRICT is an agency that administers water distribution in a large geographic region. The region is fairly arid, so the district must purchase and bring in water from outside the region. The sources of this imported water are the Colombo, Sacron, and Calorie rivers. The district then resells the water to users in the region. Its main customers are the water departments of the cities of Berdoo, Los Devils, San Go, and Hollyglass.
- It is possible to supply any of these cities with water brought in from any of the three rivers, with the exception that no provision has been made to supply Hollyglass with Calorie River water. However, because of the geographic layouts of the aqueducts and the cities in the region, the cost to the district of supplying water depends upon both the source of the water and the city being supplied. The variable cost per acre foot of water (in tens of dollars) for each combination of river and city is given in the following table. Despite these variations, the price per acre foot charged by the district is independent of the source of the water and is the same for all cities.

Water resources data for Metro Water District

	Cost (Tens of Dollars) per Acre Foot				
	Berdoo	Los Devils	San Go	Hollyglass	Supply
Colombo River	16	13	22	17	50
Sacron River	14	13	19	15	60
Calorie River	19	20	23	—	50
Minimum needed	30	70	0	10	(in units of 1
Requested	50	70	30	∞	million acre feet)

- **The management of the district is now faced with the problem of how to allocate the available water during the upcoming summer season**. In units of 1 million acre feet, the amounts available from the three rivers are given in the rightmost column of the table above. The district is committed to providing a certain minimum amount to meet the essential needs of each city (with the exception of San Go, which has an independent source of water), as shown in the *minimum needed row of the table.* The requested row indicates that Los Devils desires no more than the minimum amount, but that Berdoo would like to buy as much as 20 more, San Go would buy up to 30 more, and Hollyglass will take as much as it can get.
- Management wishes to allocate *all the available water from the three rivers to the four cities* in such a way as to at least meet the essential needs of each city while minimizing the total cost to the district.

Parameter table without minimum needs for Metro Water District

		Cost (Tens of Millions of Dollars) per Unit Distributed				
		Destination				
		Berdoo	Los Devils	San Go	Hollyglass	Supply
Source	Colombo River	16	13	22	17	50
	Sacron River	14	13	19	15	60
	Calorie River	19	20	23	*M*	50
	Dummy	0	0	0	0	50
Demand		50	70	30	60	

Parameter table for Metro Water District

			Cost (Tens of Millions of Dollars) per Unit Distributed					
			Destination					
			Berdoo (min.) 1	Berdoo (extra) 2	Los Devils 3	San Go 4	Hollyglass 5	Supply
Source	Colombo River	1	16	16	13	22	17	50
	Sacron River	2	14	14	13	19	15	60
	Calorie River	3	19	19	20	23	*M*	50
	Dummy	4(*D*)	*M*	0	*M*	0	0	50
Demand			30	20	70	30	60	

Transportation simplex method

Original simplex tableau before simplex method is applied to transportation problem

Basic Variable	Eq.	Coefficient of: Z	$\cdots$	x_{ij}	$\cdots$	z_i	$\cdots$	z_{m+j}	$\cdots$	Right side
Z	(0)	-1		c_{ij}		M		M		0
	(1)									
	$\vdots$									
z_i	(i)	0		1		1				s_i
	$\vdots$									
z_{m+j}	$(m+j)$	0		1				1		d_j
	$\vdots$									
	$(m+n)$									

Transportation simplex method

Basic Variable	Eq.	Coefficient of: Z	$\cdots$	x_{ij}	$\cdots$	z_i	$\cdots$	z_{m+j}	$\cdots$	Right Side
Z	(0)	-1		$c_{ij}-u_i-v_j$		$M-u_i$		$M-v_j$		$-\sum_{i=1}^{m} s_i u_i - \sum_{j=1}^{n} d_j v_j$

u_i = multiple of *original* row i that has been subtracted (directly or indirectly) from *original* row 0 by the simplex method during all iterations leading to the current simplex tableau.

v_j = multiple of *original* row $m + j$ that has been subtracted (directly or indirectly) from *original* row 0 by the simplex method during all iterations leading to the current simplex tableau.

- Using the duality theory, another property of the u_i *and* v_j *is that* they are the *dual variables.*
- *If* x_{ij} *is a nonbasic variable,* $c_{ij} - u_i - v_j$ *is interpreted as the* rate at which *Z will change as* x_{ij} *is increased.*

Format of a transportation simplex tableau

		Destination 1	2	$\cdots$	n	Supply	u_i
Source	1	c_{11}	c_{12}	$\cdots$	c_{1n}	s_1	
	2	c_{21}	c_{22}	$\cdots$	c_{2n}	s_2	
	$\vdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\vdots$	
	m	c_{m1}	c_{m2}	$\cdots$	c_{mn}	s_m	
Demand		d_1	d_2	$\cdots$	d_n	$Z =$	
	v_j						

Additional information to be added to each cell:

If x_{ij} is a basic variable	*If x_{ij} is a nonbasic variable*
c_{ij} ; (x_{ij}) circled	c_{ij} ; $c_{ij} - u_i - v_j$

Simplex tableau: ***m + n + 1 rows and (m + 1)(n + 1) columns***
Transportation simplex tableau: ***m rows and n columns***

Initialization

Number of basic variables $= m + n - 1.$

General Procedure for Constructing an Initial BF Solution

1. From the rows and columns still under consideration, select the next basic variable (allocation) according to some criterion.
2. Make that allocation large enough to exactly use up the remaining supply in its row or the remaining demand in its column (whichever is smaller).
3. Eliminate that row or column (whichever had the smaller remaining supply or demand) from further consideration. (If the row and column have the same remaining supply and demand, then arbitrarily select the *row* as the one to be eliminated. The column will be used later to provide a *degenerate* basic variable, i.e., a circled allocation of zero.)
4. If only one row or only one column remains under consideration, then the procedure is completed by selecting every *remaining* variable (i.e., those variables that were neither previously selected to be basic nor eliminated from consideration by eliminating their row or column) associated with that row or column to be basic with the only feasible allocation. Otherwise, return to step 1.

Criteria for Step 1

1. Northwest corner rule
2. Vogel's approximation method
3. Russell's approximation method

Northwest corner rule

Northwest corner rule: Begin by selecting x_{11} (that is, start in the northwest corner of the transportation simplex tableau). Thereafter, if x_{ij} was the last basic variable selected, then next select $x_{i,j+1}$ (that is, move one column to the *right*) if source i has any supply remaining. Otherwise, next select $x_{i+1,j}$ (that is, move one row *down*).

Example

			Cost (Tens of Millions of Dollars) per Unit Distributed					
			Destination					
			Berdoo (min.) 1	Berdoo (extra) 2	Los Devils 3	San Go 4	Hollyglass 5	Supply
Source	Colombo River	1	16	16	13	22	17	50
	Sacron River	2	14	14	13	19	15	60
	Calorie River	3	19	19	20	23	M	50
	Dummy	4(D)	M	0	M	0	0	50
Demand			30	20	70	30	60	

Example

		Destination						
		1	2	3	4	5	Supply	u_i
Source	1	16 (30)	16 (20)	13	22	17	50	
	2	14	14 (0)	13 (60)	19	15	60	
	3	19	19	20 (10)	23 (30)	M (10)	50	
	4(D)	M	0	M	0	0 (50)	50	
Demand		30	20	70	30	60	$Z = 2{,}470 + 10M$	
	v_j							

Vogel's approximation method

Vogel's approximation method: For each row and column remaining under consideration, calculate its **difference,** which is defined as *the arithmetic difference between the smallest and next-to-the-smallest unit cost c_{ij} still remaining in that row or column.* (If two unit costs tie for being the smallest remaining in a row or column, then the *difference* is 0.) In that row or column having the *largest difference,* select the variable having the *smallest remaining unit cost.* (Ties for the largest difference, or for the smallest remaining unit cost, may be broken arbitrarily.)

Example

		Destination 1	2	3	4	5	Supply	Row Difference
Source	1	16	16	13	22	17	50	3
	2	14	14	13	19	15	60	1
	3	19	19	20	23	M	50	0
	4(D)	M	0	M	[0]	0	50	0
Demand		30	20	70	30	60	Select $x_{44} = 30$	
Column difference		2	14	0	(19)	15	Eliminate column 4	

		Destination 1	2	3	5	Supply	Row Difference
Source	1	16	16	13	17	50	3
	2	14	14	13	15	60	1
	3	19	19	20	M	50	0
	4(D)	M	0	M	[0]	20	0
Demand		30	20	70	60	Select $x_{45} = 20$	
Column difference		2	14	0	(15)	Eliminate row 4(D)	

Example

		Destination 1	2	3	5	Supply	Row Difference
Source	1	16	16	[13]	17	50	(3)
	2	14	14	13	15	60	1
	3	19	19	20	M	50	0
Demand		30	20	70	40	Select $x_{13} = 50$	
Column difference		2	2	0	2	Eliminate row 1	

		Destination 1	2	3	5	Supply	Row Difference
Source	2	14	14	13	[15]	60	1
	3	19	19	20	M	50	0
Demand		30	20	20	40	Select $x_{25} = 40$	
Column difference		5	5	7	($M - 15$)	Eliminate column 5	

		Destination 1	2	3	Supply	Row Difference
Source	2	14	14	[13]	20	1
	3	19	19	20	50	0
Demand		30	20	20	Select $x_{23} = 20$	
Column difference		5	5	(7)	Eliminate row 2	

		Destination 1	2	3	Supply	
Source	3	19	19	20	50	
Demand		30	20	0	Select $x_{31} = 30$, $x_{32} = 20$, $x_{33} = 0$	$Z = 2,460$

Russell's approximation method

Russell's approximation method: For each source row i remaining under consideration, determine its $\bar{u}_i$, which is the largest unit cost c_{ij} still remaining in that row. For each destination column j remaining under consideration, determine its $\bar{v}_j$, which is the largest unit cost c_{ij} still remaining in that column. For each variable x_{ij} not previously selected in these rows and columns, calculate $\Delta_{ij} = c_{ij} - \bar{u}_i - \bar{v}_j$. Select the variable having the *largest* (in absolute terms) *negative* value of Δ_{ij}. (Ties may be broken arbitrarily.)

Example

			Cost (Tens of Millions of Dollars) per Unit Distributed					
			Destination					
			Berdoo (min.) 1	Berdoo (extra) 2	Los Devils 3	San Go 4	Hollyglass 5	Supply
Source	Colombo River	1	16	16	13	22	17	50
	Sacron River	2	14	14	13	19	15	60
	Calorie River	3	19	19	20	23	M	50
	Dummy	4(D)	M	0	M	0	0	50
Demand			30	20	70	30	60	

Example

$$\Delta_{11} = c_{11} - \bar{u}_1 - \bar{v}_1 = 16 - 22 - M = -6 - M.$$

Iteration	$\bar{u}_1$	$\bar{u}_2$	$\bar{u}_3$	$\bar{u}_4$	$\bar{v}_1$	$\bar{v}_2$	$\bar{v}_3$	$\bar{v}_4$	$\bar{v}_5$	Largest Negative Δ_{ij}	Allocation
1	22	19	M	M	M	19	M	23	M	$\Delta_{45} = -2M$	$x_{45} = 50$
2	22	19	M		19	19	20	23	M	$\Delta_{15} = -5 - M$	$x_{15} = 10$
3	22	19	23		19	19	20	23		$\Delta_{13} = -29$	$x_{13} = 40$
4		19	23		19	19	20	23		$\Delta_{23} = -26$	$x_{23} = 30$
5		19	23		19	19		23		$\Delta_{21} = -24^*$	$x_{21} = 30$
6										Irrelevant	$x_{31} = 0$
											$x_{32} = 20$
											$x_{34} = 30$
											$Z = 2{,}570$

Comparison of Alternative Criteria for Step 1

- The main virtue of the northwest corner rule is that it is quick and easy.
- However, because it pays no attention to unit costs c_{ij}, usually the solution obtained will be far from optimal.
- Vogel's approximation method has been a popular criterion for many years, partially because it is relatively easy to implement by hand.
- Because the difference represents the minimum extra unit cost incurred by failing to make an allocation to the cell having the smallest unit cost in that row or column, this criterion does take costs into account in an effective way.
- Russell's approximation method provides another excellent criterion that is still quick to implement on a computer (but not manually).
- Although it is unclear as to which is more effective on average, this criterion frequently does obtain a better solution than Vogel's.
- One distinct advantage of Russell's approximation method is that it is patterned directly after step 1 for the transportation simplex method.

Initial transportation simplex tableau from Russell's approximation method

Iteration 0		Destination 1	2	3	4	5	Supply	u_i
Source	1	16	16	13 (40)	22	17 (10)	50	
	2	14 (30)	14	13 (30)	19	15	60	
	3	19 (0)	19 (20)	20	23 (30)	M	50	
	4(D)	M	0	M	0	0 (50)	50	
Demand		30	20	70	30	60	Z = 2,570	
	v_j							

Optimality Test

Basic Variable	Eq.	Coefficient of: Z	···	x_{ij}	···	z_i	···	z_{m+j}	···	Right Side
Z	(0)	-1		$c_{ij} - u_i - v_j$		$M - u_i$		$M - v_j$		$-\sum_{i=1}^{m} s_i u_i - \sum_{j=1}^{n} d_j v_j$

- A BF solution is optimal if and only if $c_{ij} - u_i - v_j \geq 0$ *for* every *(i, j)* such that x_{ij} is nonbasic.
- Solve a system of $m + n - 1$ equations for u_i and v_j:

  $c_{ij} = u_i + v_j$ for each *(i, j)* such that x_{ij} is basic.
- Since the number of unknowns *is m + n,* a convenient choice for this purpose is to select the u_i *that has the largest number of allocations in its row (break any tie arbitrarily) and to assign to it the value zero.*

Optimality Test

x_{31}: $19 = u_3 + v_1$. Set $u_3 = 0$, so $v_1 = 19$,
x_{32}: $19 = u_3 + v_2$. $v_2 = 19$,
x_{34}: $23 = u_3 + v_4$. $v_4 = 23$.
x_{21}: $14 = u_2 + v_1$. Know $v_1 = 19$, so $u_2 = -5$.
x_{23}: $13 = u_2 + v_3$. Know $u_2 = -5$, so $v_3 = 18$.
x_{13}: $13 = u_1 + v_3$. Know $v_3 = 18$, so $u_1 = -5$.
x_{15}: $17 = u_1 + v_5$. Know $u_1 = -5$, so $v_5 = 22$.
x_{45}: $0 = u_4 + v_5$. Know $v_5 = 22$, so $u_4 = -22$.

Optimality Test

Iteration 0		Destination 1	2	3	4	5	Supply	u_i
Source	1	16 / +2	16 / +2	13 / (40)	22 / +4	17 / (10)	50	−5
	2	14 / (30)	14 / 0	13 / (30)	19 / +1	15 / −2	60	−5
	3	19 / (0)	19 / (20)	20 / +2	23 / (30)	M / M − 22	50	0
	4(D)	M / M + 3	0 / +3	M / M + 4	0 / −1	0 / (50)	50	−22
Demand		30	20	70	30	60	Z = 2,570	
	v_j	19	19	18	23	22		

An Iteration

- Step 1: Entering BV
- Step 2: Leaving BV
- Step 3: New BFS

Step 2: Leaving BV

		Destination				Supply
		...	3	4	5	
Source	1	...	13 (40)+	22 +4	17 (10)−	50
	2	...	13 (30)−	19 +1	15 [+] −2	60
		...	...	...	...	
Demand			70	30	60	

- x_{15} *becomes* the leaving basic variable.

Step 3: New BFS

		Destination				Supply
		...	3	4	5	
Source	1	...	13 (50)	22	17	50
	2	...	13 (20)	19	15 (10)	60
		...	...	...	...	
Demand			70	30	60	

Summary of Transportation Simplex Method

Initialization: Construct an initial BF solution by the procedure outlined earlier in this section. Go to the optimality test.

Optimality test: Derive u_i and v_j by selecting the row having the largest number of allocations, setting its $u_i = 0$, and then solving the set of equations $c_{ij} = u_i + v_j$ for each (i, j) such that x_{ij} is basic. If $c_{ij} - u_i - v_j \geq 0$ for every (i, j) such that x_{ij} is *nonbasic*, then the current solution is optimal, so stop. Otherwise, go to an iteration.

Iteration:

1. Determine the entering basic variable: Select the nonbasic variable x_{ij} having the *largest* (in absolute terms) *negative* value of $c_{ij} - u_i - v_j$.
2. Determine the leaving basic variable: Identify the chain reaction required to retain feasibility when the entering basic variable is increased. From the donor cells, select the basic variable having the *smallest* value.
3. Determine the new BF solution: Add the value of the leaving basic variable to the allocation for each recipient cell. Subtract this value from the allocation for each donor cell.

Complete set of transportation simplex tableaux for Metro Water District problem

Iteration 0		Destination 1	2	3	4	5	Supply	u_i
Source	1	16 +2	16 +2	13 (40)+	22 +4	17 (10)−	50	−5
	2	14 (30)	14 0	13 (30)−	19 +1	15 [+] −2	60	−5
	3	19 (0)	19 (20)	20 +2	23 (30)	M M − 22	50	0
	4(D)	M M + 3	0 +3	M M + 4	0 −1	0 (50)	50	−22
Demand		30	20	70	30	60	Z = 2,570	
	v_j	19	19	18	23	22		

Complete set of transportation simplex tableaux for Metro Water District problem

Iteration 1		Destination 1	Destination 2	Destination 3	Destination 4	Destination 5	Supply	u_i
Source	1	16; +2	16; +2	13; (50)	22; +4	17; +2	50	−5
	2	14; (30)−	14; 0	13; (20)	19; +1	15; (10)+	60	−5
	3	19; (0)+	19; (20)	20; +2	23; (30)−	M; M − 20	50	0
	4(D)	M; M + 1	0; +1	M; M + 2	0; [+] −3	0; (50)−	50	−20
Demand		30	20	70	30	60	Z = 2,550	
	v_j	19	19	18	23	20		

Complete set of transportation simplex tableaux for Metro Water District problem

Iteration 2		Destination 1	Destination 2	Destination 3	Destination 4	Destination 5	Supply	u_i
Source	1	16; +5	16; +5	13; (50)	22; +7	17; +2	50	−8
	2	14; +3	14; +3	13; (20)−	19; +4	15; (40)+	60	−8
	3	19; (30)	19; (20)	20; [+] −1	23; (0)−	M; M − 23	50	0
	4(D)	M; M + 4	0; +4	M; M + 2	0; (30)+	0; (20)−	50	−23
Demand		30	20	70	30	60	Z = 2,460	
	v_j	19	19	21	23	23		

Complete set of transportation simplex tableaux for Metro Water District problem

Iteration 3		Destination 1	Destination 2	Destination 3	Destination 4	Destination 5	Supply	u_i
Source	1	16; +4	16; +4	13; (50)	22; +7	17; +2	50	−7
	2	14; +2	14; +2	13; (20)	19; +4	15; (40)	60	−7
	3	19; (30)	19; (20)	20; (0)	23; +1	M; M − 22	50	0
	4(D)	M; M + 3	0; +3	M; M + 2	0; (30)	0; (20)	50	−22
Demand		30	20	70	30	60	Z = 2,460	
	v_j	19	19	20	22	22		

Practice Problem

8.1-6. The Onenote Co. produces a single product at three plants for four customers. The three plants will produce 60, 80, and 40 units, respectively, during the next time period. The firm has made a commitment to sell 40 units to customer 1, 60 units to customer 2, and at least 20 units to customer 3. Both customers 3 and 4 also want to buy as many of the remaining units as possible. The net profit associated with shipping a unit from plant i for sale to customer j is given by the following table:

		Customer 1	Customer 2	Customer 3	Customer 4
Plant	1	$800	$700	$500	$200
	2	$500	$200	$100	$300
	3	$600	$400	$300	$500

Management wishes to know how many units to sell to customers 3 and 4 and how many units to ship from each of the plants to each of the customers to maximize profit.

(a) Formulate this problem as a transportation problem where the objective function is to be maximized by constructing the appropriate parameter table that gives unit profits.

Assignment Problem

- The **assignment problem is a special type of linear programming problem where assignees** are being assigned to perform **tasks.**
- For example, the assignees might be employees who need to be given work assignments.
- Assigning people to jobs is a common application of the assignment problem.
- However, the assignees need not be people. They also could be machines, or vehicles, or plants, or even time slots to be assigned tasks.

Assignment Problem

- The assignment problem can be viewed as a special type of transportation problem.
- An **assignment problem is a balanced transportation** problem in which all supplies and demands are equal to 1.
- An assignment problem is characterized by knowledge of the cost of assigning each supply point to each demand point.
- The assignment problem's matrix of costs is its **cost matrix.**

Assignment Problem Assumptions

1. The number of assignees and the number of tasks are the same. (This number is denoted by *n.)*
2. Each assignee is to be assigned to exactly *one task.*
3. Each task is to be performed by exactly *one assignee.*
4. There is a cost c_{ij} *associated with assignee i (i 1, 2, . . . , n) performing task j (j 1, 2, . . . , n).*
5. The objective is to determine how all *n assignments should be made to minimize the* total cost.

Any problem satisfying all these assumptions can be solved extremely efficiently by algorithms designed specifically for assignment problems.

Assignment Problem: Example1

- The JOB SHOP COMPANY has purchased three new machines of different types. There are four available locations in the shop where a machine could be installed. Some of these locations are more desirable than others for particular machines because of their proximity to work centers that will have a heavy work flow to and from these machines. (There will be no work flow *between the new machines.) Therefore, the objective is to assign the* new machines to the available locations to minimize the total cost of materials handling. The estimated cost in dollars per hour of materials handling involving each of the machines is given in the following table for the respective locations. Location 2 is not considered suitable for machine 2, so no cost is given for this case.

Materials-handling cost data ($) for Job Shop Co.

		Location			
		1	2	3	4
Machine	1	13	16	12	11
	2	15	—	13	20
	3	5	7	10	6

Cost table for Job Shop Co. assignment problem

		Task (Location)			
		1	2	3	4
Assignee (Machine)	1	13	16	12	11
	2	15	*M*	13	20
	3	5	7	10	6
	4(*D*)	0	0	0	0

Assignment Problem Model

- Assignment problems involve "yes/no" decisions.
- For example: Should assignee *i* perform task *j*?
- Decision variables: $x_{ij} = \begin{cases} 1 & \text{if assignee } i \text{ performs task } j, \\ 0 & \text{if not,} \end{cases}$

Assignment problem model

$$\text{Minimize} \quad Z = \sum_{i=1}^{n} \sum_{j=1}^{n} c_{ij} x_{ij},$$

subject to

$$\sum_{j=1}^{n} x_{ij} = 1 \quad \text{for } i = 1, 2, \ldots, n,$$

$$\sum_{i=1}^{n} x_{ij} = 1 \quad \text{for } j = 1, 2, \ldots, n,$$

and

$$x_{ij} \geq 0, \quad \text{for all } i \text{ and } j$$
$$(x_{ij} \text{ binary}, \quad \text{for all } i \text{ and } j).$$

Comparing Transportation & Assignment Problems

- Assignment problem is just a special type of transportation problem where *the sources now are assignees and the destinations now are tasks* and where
- Number of sources *m = number of destinations n,*
- Every supply $s_i = 1$,
- Every demand $d_j = 1$.

Assignment Problem: Example 2

- The BETTER PRODUCTS COMPANY has decided to initiate the production of four new products, using three plants that currently have excess production capacity. The products require a comparable production effort per unit, so the available production capacity of the plants is measured by the number of units of any product that can be produced per day, as given in the rightmost column of the table below. The bottom row gives the required production rate per day to meet projected sales. Each plant can produce any of these products, *except that Plant 2 cannot produce product 3. However, the variable costs per unit* of each product differ from plant to plant, as shown in the main body of the table.

		Unit Cost ($) for Product				Capacity Available
		1	2	3	4	
Plant	1	41	27	28	24	75
	2	40	29	—	23	75
	3	37	30	27	21	45
Production rate		20	30	30	40	

Assignment Problem: Example 2

- Management now needs to make a decision on how to split up the production of the products among plants. Two kinds of options are available.
- *Option 1: Permit product splitting, where the same product is produced in more than one* plant.
- *Option 2: Prohibit product splitting.* For Option 2, management further specifies that every plant should be assigned at least one of the products.

Formulation of Option 1

- **Parameter table for the transportation problem formulation of Option 1 for the Better Products Co. problem**

		Cost per Unit Distributed					
		Destination (Product)					
		1	2	3	4	5(*D*)	Supply
Source (Plant)	1	41	27	28	24	0	75
	2	40	29	*M*	23	0	75
	3	37	30	27	21	0	45
Demand		20	30	30	40	75	

Formulation of Option 2

- Without product splitting, each product must be assigned to just one plant. Therefore, producing the products can be interpreted as the tasks for an assignment problem, where the plants are the assignees.
- Management has specified that every plant should be assigned at least one of the products. There are more products (four) than plants (three), so one of the plants will need to be assigned two products. Plant 3 has only enough excess capacity to produce one product, so *either Plant 1 or Plant 2 will take the extra product.*

Cost table for assignment problem formulation of Option 2 for Better Products Co. problem

		Task (Product)				
		1	2	3	4	5(*D*)
Assignee (Plant)	1*a*	820	810	840	960	0
	1*b*	820	810	840	960	0
	2*a*	800	870	*M*	920	0
	2*b*	800	870	*M*	920	0
	3	740	900	810	840	*M*

Hungarian Method

- If there are to be *n* matches, there are *n*! possibilities.
- A much simpler approach is Hungarian method (Kuhn, 1955; Munkres, 1957) → method for assigning jobs by a one-for-one matching to identify the lowest cost solution.
- The Hungarian method is a combinatorial optimization algorithm which finds an optimal assignment for a given cost matrix in polynomial time.
- It was originally proposed by H.W. Kuhn in 1955, who gave the name *Hungarian method,* because the algorithm was based on the earlier work of two Hungarian mathematicians.
- The algorithm was later refined by J. Munkres in 1957.

Hungarian method

1. Subtract the smallest number in each row from every number in the row. This is called a *row reduction*. Enter the results in a new table.
2. Subtract the smallest number in each column from every number in the column. This is called a column *reduction*. Enter the results in another table.
3. Test whether an optimum assignment can be made. You do this by determining the *minimum* number of lines (horizontal or vertical) needed to cross out all zeros. If the number of lines equals the number of rows, an optimum assignment is possible. In that case, go to step 6. Otherwise go on to step 4.
4. If the number of lines is less than the number of rows, modify the table in this way:
   a) Subtract the smallest uncovered number from every uncovered number in the table.
   b) Add the smallest uncovered number to the numbers at *intersections* of cross-out lines.
   c) Numbers crossed out but not at intersections of cross-out lines carry over unchanged to the next table.
5. Repeat steps 3 and 4 until an optimal table is obtained.
6. Make the assignments. Begin with rows or columns with only one zero. Match items that have zeros, using only one match for each row and each column. Eliminate both the row and the column after the match

Example: Hungarian Method

- Determine the optimum assignment of jobs to machines for the following data.

		MACHINE			
		A	B	C	D
Job	1	8	6	2	4
	2	6	7	11	10
	3	3	5	7	6
	4	5	10	12	9