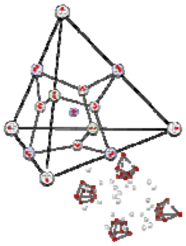
Lecture 10: The Simplex Method



Outline

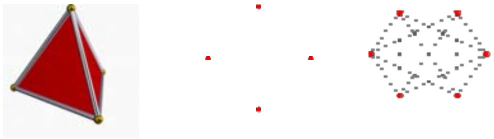
- Essence of the Simplex Method
- Solution Concepts
- Setting up the Simplex Method

The Simplex

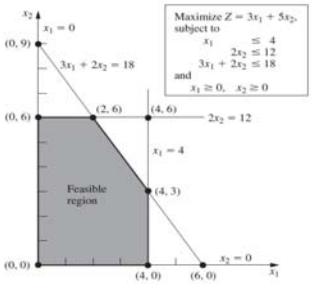
- A general procedure for solving linear programming problems.
- Extensions and variations of the simplex method also are used to perform *postoptimality analysis* (*including sensitivity* analysis) on the model.

The Essence of the Simplex

- The simplex method is an algebraic procedure.
- However, its underlying concepts are geometric.
- In geometry, a **simplex** is a generalization of the notion of a triangle or tetrahedron to arbitrary dimensions.



Wyndor Problem



- A **constraint boundary** is a **line** that forms the boundary of what is permitted by the corresponding constraint.
- The points of intersection are the **corner-point solutions** of the problem.

Corner Point Feasible Solutions

- Corner point feasible solutions (CPF solutions) occur at the intersections of the constraint boundaries, which are in the feasible region.

- The problem has 2 decision variables, so each corner occurs at the intersection of two constraints.

- The dimension of the problem determines the number of constraints that intersect at each corner.

- If there are, for example, 5 decision variables the corner points occur at the intersection of 5 constraints.

Adjacent feasible solutions

- For any linear programming model with *n* decision variables, two CPF solutions are adjacent to each other if they share *n*-1 constraints.

- The two adjacent CPF solutions are connected by a line segment that lies on the same shared constraint boundaries.

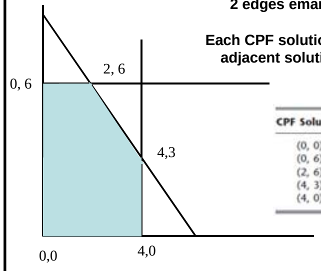
- Such a constraint boundary is referred to as an edge of the feasible region.

Wyndor Problem

There are 5 edges.

**From each CPF solution,
2 edges emanate.**

**Each CPF solution has 2
adjacent solutions.**



CPF Solution	Its Adjacent CPF Solutions
(0, 0)	(0, 6) and (4, 0)
(0, 6)	(2, 6) and (0, 0)
(2, 6)	(4, 3) and (0, 6)
(4, 3)	(4, 0) and (2, 6)
(4, 0)	(0, 0) and (4, 3)

Optimality test

If a CPF solution has no adjacent CPF solution that is better (in terms of the value of the objective function) then it must be an optimal solution.

Solving the example

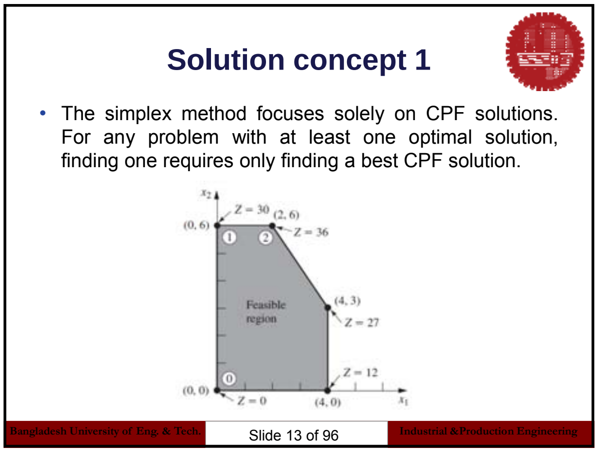
- Initialization. Choose point (0,0) because it is convenient (no calculation needed to show it is feasible)
- Optimality test: Conclude that (0,0) is not optimal because adjacent CPF solutions are better
- Iteration 1: Move to a better CPF solution

Simplex Iteration

- Consider the two edges that emanate from (0,0). Move up the X_2 axis because X_2 contributes more to the objective function of $Z = 3X_1 + 5X_2$.
- Stop at the next constraint boundary, $2X_2 = 12$ (or $X_2 = 6$). This is point (0,6). You can't move further in this direction without leaving the feasible region.
- Find the other adjacent CPF solution for this point (2,6) from the intersection of constraints.
- Optimality test: (0,6) is not optimal because (2,6) is better.
- Repeat steps. (2,6) is optimal solution.

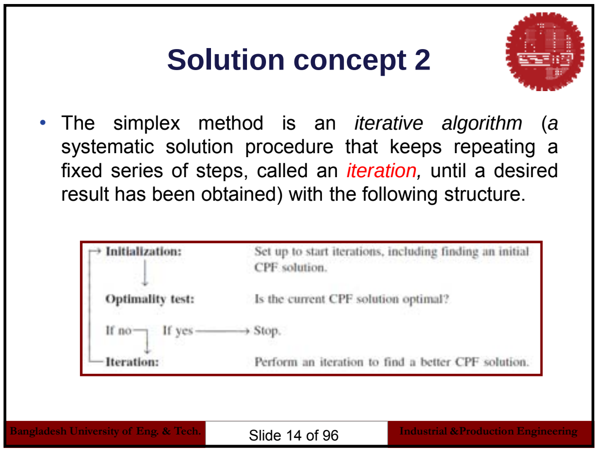
The Key Solution Concepts

- Simplex focuses only on CPF solutions, a finite set.
- It is an iterative procedure, meaning a fixed series of steps is repeated (an iteration) until a desired result is found.
- When possible, the initial CPF solution is the origin because it is convenient. (Possible when all variables are non-negative.) If origin is infeasible, special procedures are needed, will be discussed later.



Solution concept 1

- The simplex method focuses solely on CPF solutions. For any problem with at least one optimal solution, finding one requires only finding a best CPF solution.



Solution concept 2

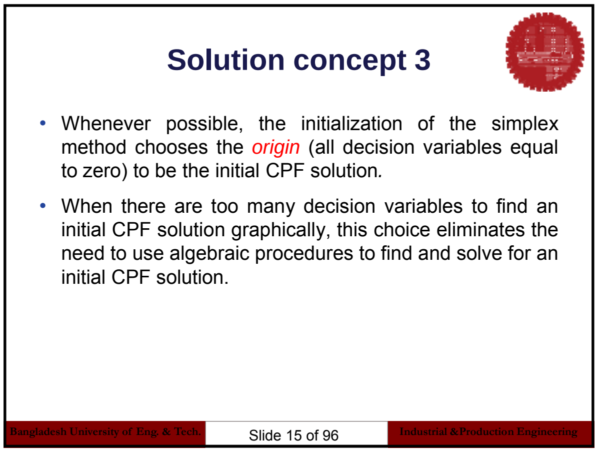
- The simplex method is an *iterative algorithm* (a systematic solution procedure that keeps repeating a fixed series of steps, called an *iteration*, until a desired result has been obtained) with the following structure.

Initialization: Set up to start iterations, including finding an initial CPF solution.

Optimality test: Is the current CPF solution optimal?

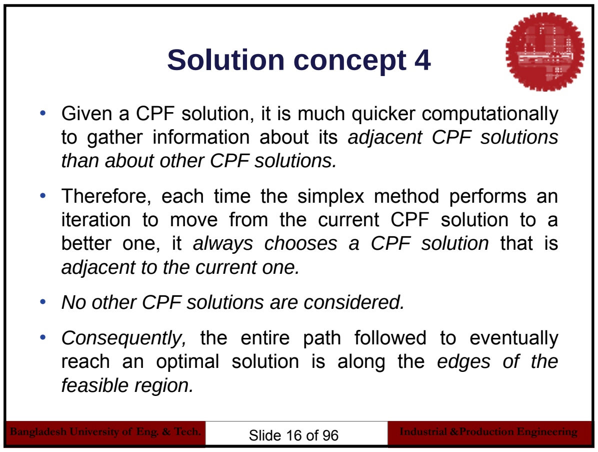
If no — If yes ⟶ Stop.

Iteration: Perform an iteration to find a better CPF solution.



Solution concept 3

- Whenever possible, the initialization of the simplex method chooses the *origin* (all decision variables equal to zero) to be the initial CPF solution.
- When there are too many decision variables to find an initial CPF solution graphically, this choice eliminates the need to use algebraic procedures to find and solve for an initial CPF solution.



Solution concept 4

- Given a CPF solution, it is much quicker computationally to gather information about its *adjacent CPF solutions than about other CPF solutions.*
- Therefore, each time the simplex method performs an iteration to move from the current CPF solution to a better one, it *always chooses a CPF solution* that is *adjacent to the current one.*
- *No other CPF solutions are considered.*
- *Consequently,* the entire path followed to eventually reach an optimal solution is along the *edges of the feasible region.*

Solution concept 5

- After the current CPF solution is identified, the simplex method examines each of the edges of the feasible region that emanate from this CPF solution.
- Each of these edges leads to an *adjacent CPF solution at the other* end, but the simplex method does not even take the time to solve for the adjacent CPF solution.
- Instead, it simply identifies the *rate of improvement in Z that* would be obtained by moving along the edge.
- Among the edges with a *positive* rate of improvement in *Z, it then chooses to move along the one with the largest* rate of improvement in *Z.*
- The iteration is completed by first solving for the adjacent CPF solution at the other end of this one edge and *then relabeling this adjacent CPF solution as the current CPF* solution for the optimality test and (if needed) the next iteration.

Solution concept 6

- The *optimality test* consists simply of checking whether any of the edges give a positive rate of improvement in *Z.*
- If none do, then the current CPF solution is optimal.

Setting up the Simplex Method

• We convert the inequalities to equalities by use of slack variables.

Let X_3 be a slack for the first constraint

$X_1 \leq 4$ becomes $X_1 + X_3 = 4$

With $X_3 \geq 0$

Setting up the Simplex Method

Original Form of the Model

Maximize $Z = 3x_1 + 5x_2,$

subject to

$$\begin{aligned} x_1 &\leq 4 \\ 2x_2 &\leq 12 \\ 3x_1 + 2x_2 &\leq 18 \end{aligned}$$

and

$$x_1 \geq 0, \quad x_2 \geq 0.$$

Augmented Form of the Model[1]

Maximize $Z = 3x_1 + 5x_2,$

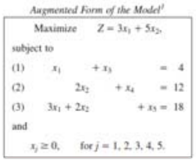
subject to

$$\begin{aligned} &(1) \quad x_1 + x_3 = 4 \\ &(2) \quad 2x_2 + x_4 = 12 \\ &(3) \quad 3x_1 + 2x_2 + x_5 = 18 \end{aligned}$$

and

$$x_j \geq 0, \quad \text{for } j = 1, 2, 3, 4, 5.$$

Augmented Solution



Augmented Form of the Model[1]

$$\text{Maximize} \quad Z = 3x_1 + 5x_2,$$

subject to

$$
\begin{array}{llllll}
(1) & x_1 & & +x_3 & & = 4 \\
(2) & & 2x_2 & & +x_4 & = 12 \\
(3) & 3x_1 & +2x_2 & & +x_5 & = 18
\end{array}
$$

and

$$x_j \geq 0, \quad \text{for } j = 1, 2, 3, 4, 5.$$

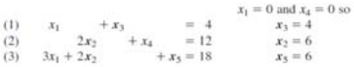
- The augmented solution contains values for the decision variables AND values for the slacks.
- So if we augment the solution (3, 2) in this example by the values of the slacks we get (3, 2, 1, 8, 5).
- A *basic solution* is an augmented corner-point solution.

Properties of a basic solution

- Each variable is either basic or non-basic.
- The number of basic variables equals the number of functional constraints.
- The number of non-basic variables equals the total number of variables minus the number of functional constraints.
- The non-basic variables are set equal to zero.
- The values of the basic variables are found by solving the simultaneous equation system.
- If the basic solution satisfies the non-negativity consideration, it is called a basic feasible solution.

Example: BF Solution

(0, 6, 4, 0, 6)



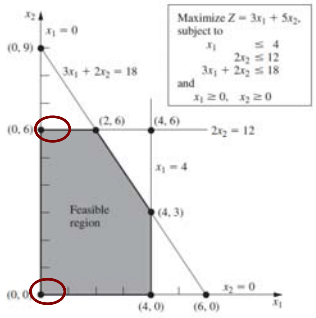
$x_1 = 0$ and $x_4 = 0$ so

$$
\begin{array}{llllll}
(1) & x_1 & & +x_3 & & = 4 \\
(2) & & 2x_2 & & +x_4 & = 12 \\
(3) & 3x_1 & +2x_2 & & +x_5 & = 18
\end{array}
\qquad
\begin{array}{l}
x_3 = 4 \\
x_2 = 6 \\
x_5 = 6
\end{array}
$$

Test for adjacent points

- Two BF solutions are adjacent if all but one of their non-basic variables are the same.
- All but one of the basic variables would correspond in the two points, but perhaps with different values.
- Moving to an adjacent point means we swap one basic variable for a non-basic one. One variable enters; the other leaves.

Test for adjacent points



Algebra of the Simplex

Maximize Z,

subject to

$$
\begin{array}{llr}
(0) & Z - 3x_1 - 5x_2 & = 0 \\
(1) & x_1 \quad + x_3 & = 4 \\
(2) & 2x_2 \quad + x_4 & = 12 \\
(3) & 3x_1 + 2x_2 \quad + x_5 & = 18
\end{array}
$$

and

$x_j \geq 0, \quad$ for $j = 1, 2, \ldots, 5.$

Algebra of the Simplex

Initialization: $X_1 = 0$ and $X_2 = 0$

Yields: $X_3 = 4$, $X_4 = 12$, and $X_5 = 18$

$$
\begin{array}{llr}
(1) & x_1 \quad + x_3 & = 4 \\
(2) & 2x_2 \quad + x_4 & = 12 \\
(3) & 3x_1 + 2x_2 \quad + x_5 & = 18
\end{array}
$$

- Notice that the BFS can be read immediately from RHS.
- The simplex used a procedure (Gaussian elimination) to convert the equations to the same convenient form with each iteration.

Geometric and algebraic interpretations

Geometric	Algebraic
Choose (0,0) as initial CPF solution.	Choose $X_1$and X_2 to be non-basic for initial BFS (0,0,4,12,18)
Optimality test: not optimal because moving along either edge increases *Z*.	Not optimal because increasing either non-basic variable increases *Z*.
Iteration 1, step 1: Move up the edge lying on the X_2 axis.	Iteration 1, step 1: Increase X_2 while adjusting other variable values to satisfy the system of equations.

Geometric and algebraic interpretations

Geometric	Algebraic
Iteration 1, step 2; Stop when the first new constraint boundary (X_2=6) is reached.	Iteration 1, step 2: Stop when the first basic variable (X_3, X_4, or X_5) drops to zero.
Iteration 1, step 3: Find the intersection of the new pair of constraint boundaries: (0,6) is the new CPF solution.	Iteration 1, step 3. With X_2 now basic and X_4 non-basic, solve the system of equations to find the new BFS (0,6,4,0,6)

Geometric and algebraic interpretations

Geometric	Algebraic
Optimality test: Not optimal because moving along the edge from (0,6) to the right increases Z.	Optimality test: Not optimal because increasing one non-basic variable (X_1) increases Z.
Iteration 2: Move along the edge to the right and stop when the new constraint boundary is reached.	Iteration 2: Increase X_1, while adjusting other variables and stop when the first basic variable reaches 0.
Optimality test : (optimal)	Optimality test: (optimal)

Direction of Movement

$Z = 3X_1 + 5X_2$

If we increase X_1 rate of improvement is 3.
If we increase X_2 rate of improvement is 5.

Choose X_2.

X_2 is the entering basic variable for iteration 1.

Where to stop?

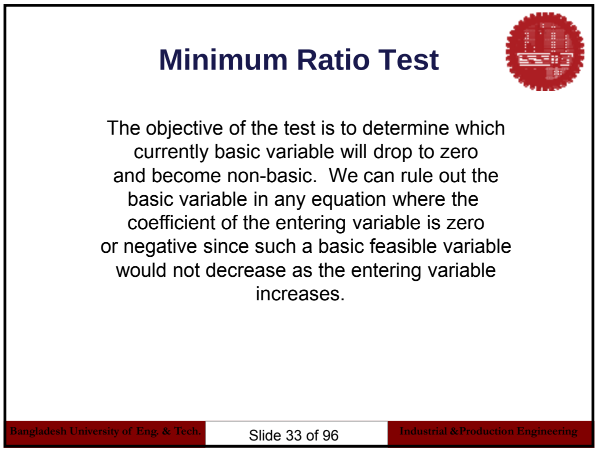
Increasing X_2 increases Z. We want to go as far as possible without leaving the feasible region.

$$\begin{array}{lllll} (1) & x_1 & & + x_3 & = 4 \\ (2) & & 2x_2 & + x_4 & = 12 \\ (3) & 3x_1 + 2x_2 & & + x_5 & = 18 \end{array}$$

Check the constraints

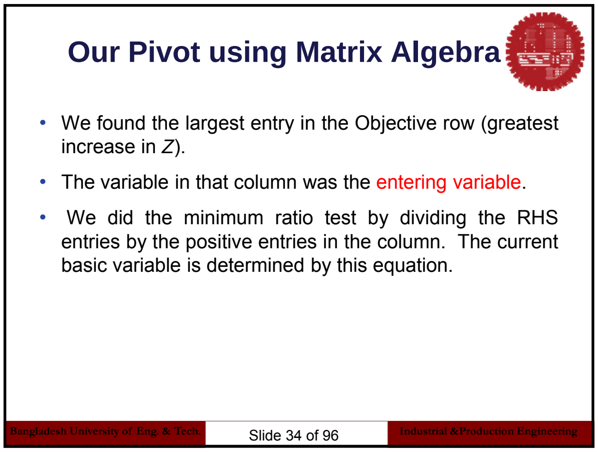
$X_3 = 4 \geq 0$ No upper bound on X_2
$X_4 = 12 - 2X_2 \geq 0$ $X_2 \leq 6$ to keep X_4 non-neg
$X_5 = 18 - 2X_2 \geq 0$ $X_2 \leq 9$ to keep X_5 non-neg

Thus, X_2 can be increased to 6, at which point X_4 drops to zero (non-basic). This is the minimum ratio test.



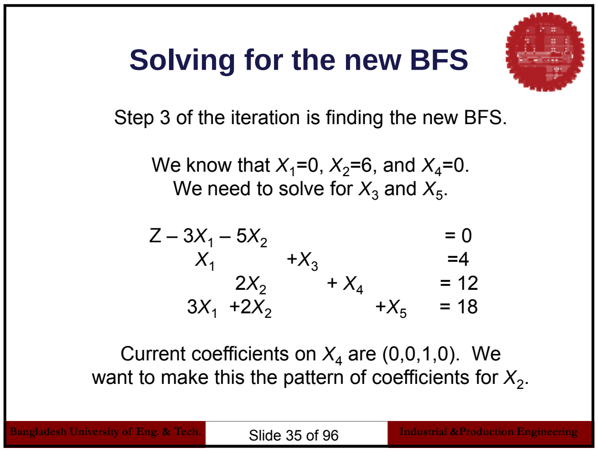
Minimum Ratio Test

The objective of the test is to determine which currently basic variable will drop to zero and become non-basic. We can rule out the basic variable in any equation where the coefficient of the entering variable is zero or negative since such a basic feasible variable would not decrease as the entering variable increases.



Our Pivot using Matrix Algebra

- We found the largest entry in the Objective row (greatest increase in Z).
- The variable in that column was the entering variable.
- We did the minimum ratio test by dividing the RHS entries by the positive entries in the column. The current basic variable is determined by this equation.



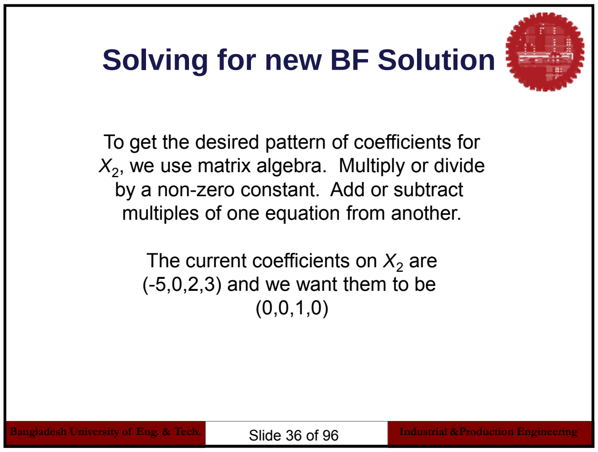
Solving for the new BFS

Step 3 of the iteration is finding the new BFS.

We know that X_1=0, X_2=6, and X_4=0.
We need to solve for X_3 and X_5.

$$\begin{aligned} Z - 3X_1 - 5X_2 &= 0 \\ X_1 \quad + X_3 &= 4 \\ 2X_2 \quad + X_4 &= 12 \\ 3X_1 + 2X_2 \quad + X_5 &= 18 \end{aligned}$$

Current coefficients on X_4 are (0,0,1,0). We want to make this the pattern of coefficients for X_2.



Solving for new BF Solution

To get the desired pattern of coefficients for X_2, we use matrix algebra. Multiply or divide by a non-zero constant. Add or subtract multiples of one equation from another.

The current coefficients on X_2 are (-5,0,2,3) and we want them to be (0,0,1,0)

Transformed Equations

$$\begin{array}{llllll}
Z - 3X_1 & & & + 5/2X_4 & & = 30 \\
X_1 & & +X_3 & & & = 4 \\
 & X_2 & & + ½\, X_4 & & = 6 \\
3X_1 & & & - X_4 & + X_5 & = 6
\end{array}$$

Since $X_1 = 0$ and $X_4 = 0$, the equations in this form give the BFS (0, 6, 4, 0, 6)

Optimality test for new BF solution

The current objective function gives the value of Z in terms of only the current nonbasic variables.

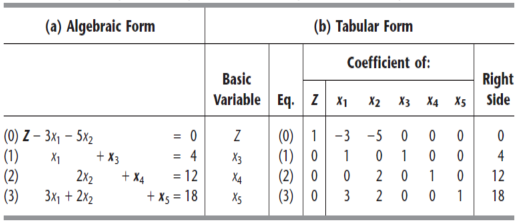
$Z = 30 + 3X_1 – 5/2\, X_4$

If X_1 increases, Z will increase. In iteration 2, we let X_1 enter and find a variable to leave. Or minimum ratio test indicates X_5 will leave. We repeat the algebraic manipulations to reproduce the pattern of coefficients for X_5 (0,0,0,1) as the new coefficients of X_1.

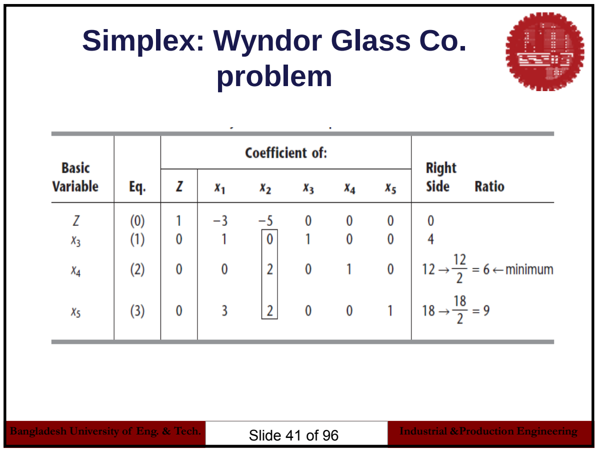
Tabular form

- It provides a summary of important information
- The simplex tableau is a compact display of the system of equations

Initial system of equations for Wyndor Glass Co. problem

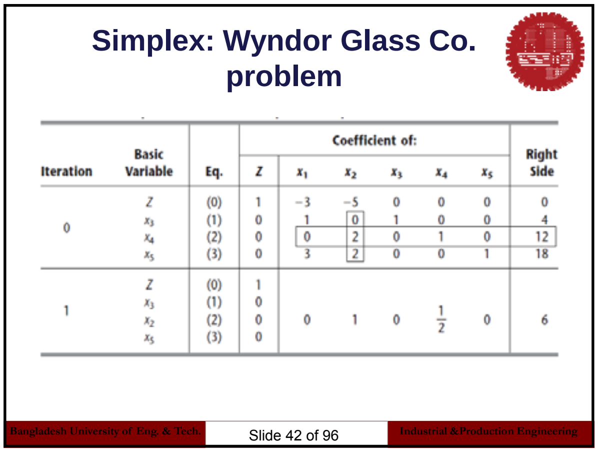


(a) Algebraic Form	(b) Tabular Form								
	Basic Variable	Eq.	Coefficient of:						Right Side
			Z	x_1	x_2	x_3	x_4	x_5	
(0) $Z - 3x_1 - 5x_2 = 0$	Z	(0)	1	−3	−5	0	0	0	0
(1) $x_1 + x_3 = 4$	x_3	(1)	0	1	0	1	0	0	4
(2) $2x_2 + x_4 = 12$	x_4	(2)	0	0	2	0	1	0	12
(3) $3x_1 + 2x_2 + x_5 = 18$	x_5	(3)	0	3	2	0	0	1	18



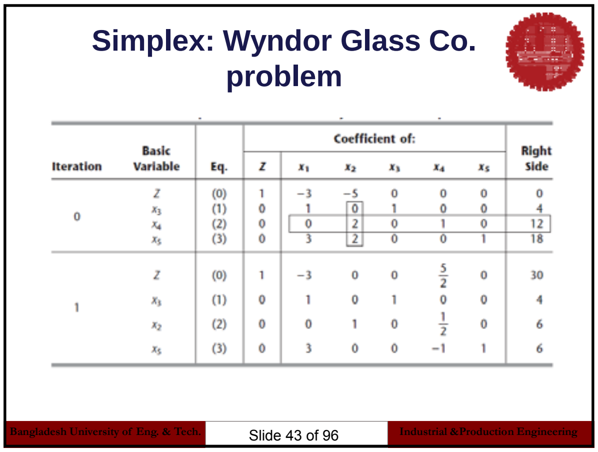
Simplex: Wyndor Glass Co. problem

Basic Variable	Eq.	Coefficient of: Z	x_1	x_2	x_3	x_4	x_5	Right Side	Ratio
Z	(0)	1	−3	−5	0	0	0	0	
x_3	(1)	0	1	0	1	0	0	4	
x_4	(2)	0	0	2	0	1	0	$12 \rightarrow \frac{12}{2} = 6 \leftarrow$ minimum	
x_5	(3)	0	3	2	0	0	1	$18 \rightarrow \frac{18}{2} = 9$	



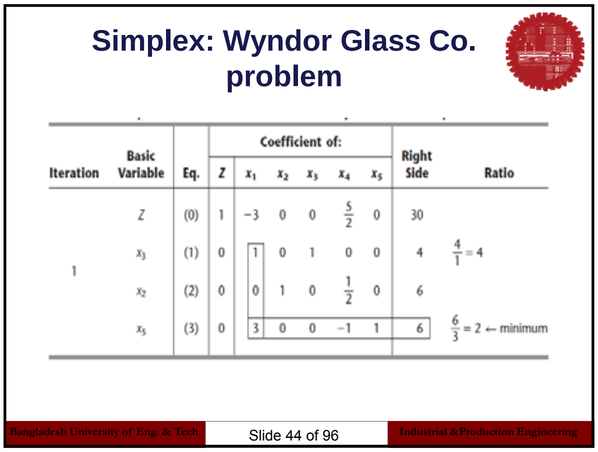
Simplex: Wyndor Glass Co. problem

Iteration	Basic Variable	Eq.	Coefficient of: Z	x_1	x_2	x_3	x_4	x_5	Right Side
0	Z	(0)	1	−3	−5	0	0	0	0
	x_3	(1)	0	1	0	1	0	0	4
	x_4	(2)	0	0	2	0	1	0	12
	x_5	(3)	0	3	2	0	0	1	18
1	Z	(0)	1						
	x_3	(1)	0						
	x_2	(2)	0	0	1	0	$\frac{1}{2}$	0	6
	x_5	(3)	0						



Simplex: Wyndor Glass Co. problem

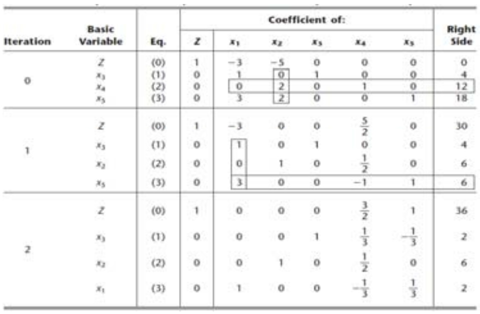
Iteration	Basic Variable	Eq.	Coefficient of: Z	x_1	x_2	x_3	x_4	x_5	Right Side
0	Z	(0)	1	−3	−5	0	0	0	0
	x_3	(1)	0	1	0	1	0	0	4
	x_4	(2)	0	0	2	0	1	0	12
	x_5	(3)	0	3	2	0	0	1	18
1	Z	(0)	1	−3	0	0	$\frac{5}{2}$	0	30
	x_3	(1)	0	1	0	1	0	0	4
	x_2	(2)	0	0	1	0	$\frac{1}{2}$	0	6
	x_5	(3)	0	3	0	0	−1	1	6



Simplex: Wyndor Glass Co. problem

Iteration	Basic Variable	Eq.	Coefficient of: Z	x_1	x_2	x_3	x_4	x_5	Right Side	Ratio
1	Z	(0)	1	−3	0	0	$\frac{5}{2}$	0	30	
	x_3	(1)	0	1	0	1	0	0	4	$\frac{4}{1} = 4$
	x_2	(2)	0	0	1	0	$\frac{1}{2}$	0	6	
	x_5	(3)	0	3	0	0	−1	1	6	$\frac{6}{3} = 2 \leftarrow$ minimum

Complete set of simplex tableaux for the Wyndor Glass Co. problem



Iteration	Basic Variable	Eq.	Coefficient of:						Right Side
			Z	x_1	x_2	x_3	x_4	x_5	
0	Z	(0)	1	−3	−5	0	0	0	0
	x_3	(1)	0	1	$\boxed{0}$	1	0	0	4
	x_4	(2)	0	0	$\boxed{2}$	0	1	0	12
	x_5	(3)	0	3	$\boxed{2}$	0	0	1	18
1	Z	(0)	1	−3	0	0	$\frac{5}{2}$	0	30
	x_3	(1)	0	$\boxed{1}$	0	1	0	0	4
	x_2	(2)	0	$\boxed{0}$	1	0	$\frac{1}{2}$	0	6
	x_5	(3)	0	$\boxed{3}$	0	0	−1	1	6
2	Z	(0)	1	0	0	0	$\frac{3}{2}$	1	36
	x_3	(1)	0	0	0	1	$\frac{1}{3}$	$-\frac{1}{3}$	2
	x_2	(2)	0	0	1	0	$\frac{1}{2}$	0	6
	x_1	(3)	0	1	0	0	$-\frac{1}{3}$	$\frac{1}{3}$	2

Tie Breaking

- Tie for entering variable: If two variables are tied for largest negative value in obj row, pick one arbitrarily. No problems should result.
- Tie for leaving basic variable (degeneracy): This is a tie for the minimum ratio. Whichever variable is picked to leave, the other variable will also be driven to zero in the pivot. Problems may ensue.

Degeneracy

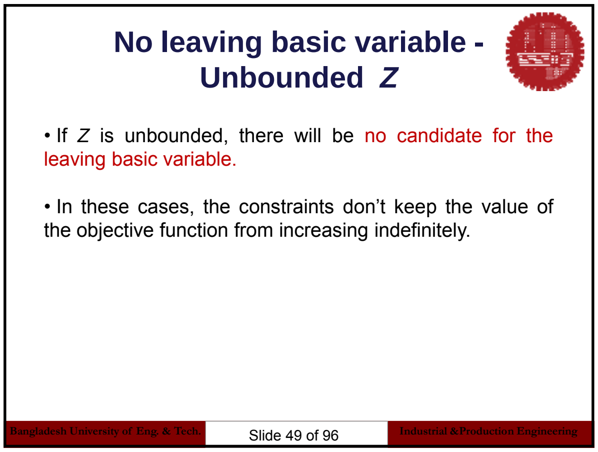
• If one of the degenerate basic variables (basic variables with a value of zero) retains its zero value until it is chosen at a subsequent iteration to be the leaving basic variable, the corresponding entering variable will be stuck at zero since it can't be increased without making the degenerate leaving variable negative, so the value of Z won't change.

•Simplex may go around in a loop, repeating the same sequence without advancing.

Degeneracy (cont'd)

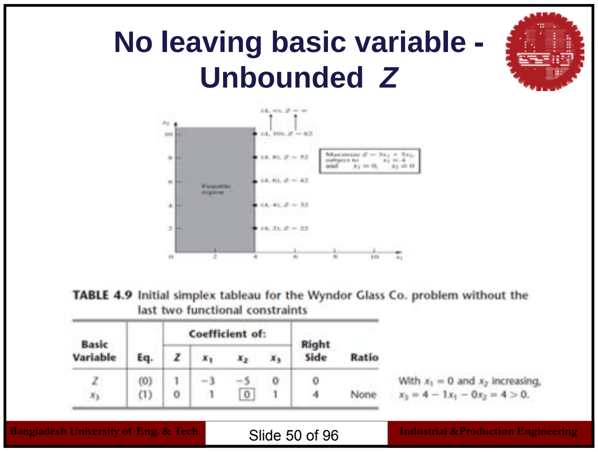
• Perpetual loops are rare. When they do occur, you can get out of it by picking a different leaving basic variable from the tie.
• Special rules have been developed for tie breaking so that the loops are avoided.

See: R. Bland "New Finite Pivoting Rules for the Simplex Method." Mathematics of Operations Research, 2: 103-107, 1977.



No leaving basic variable - Unbounded *Z*

- If *Z* is unbounded, there will be no candidate for the leaving basic variable.
- In these cases, the constraints don't keep the value of the objective function from increasing indefinitely.

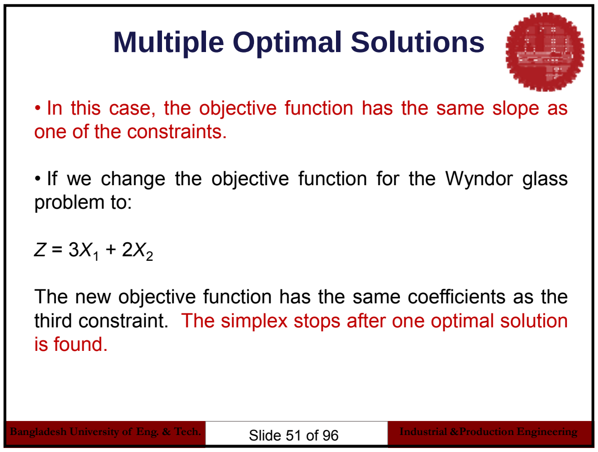


No leaving basic variable - Unbounded *Z*

TABLE 4.9 Initial simplex tableau for the Wyndor Glass Co. problem without the last two functional constraints

Basic Variable	Eq.	Coefficient of: Z	x_1	x_2	x_3	Right Side	Ratio
Z	(0)	1	−3	−5	0	0	
x_3	(1)	0	1	$\boxed{0}$	1	4	None

With $x_1 = 0$ and x_2 increasing, $x_3 = 4 - 1x_1 - 0x_2 = 4 > 0$.

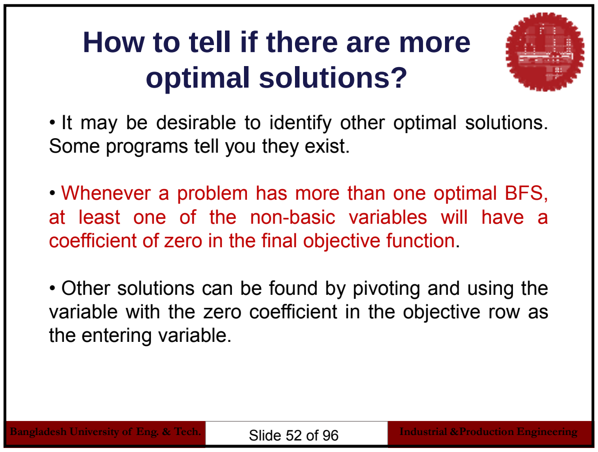


Multiple Optimal Solutions

- In this case, the objective function has the same slope as one of the constraints.
- If we change the objective function for the Wyndor glass problem to:

$Z = 3X_1 + 2X_2$

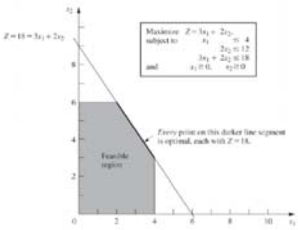
The new objective function has the same coefficients as the third constraint. The simplex stops after one optimal solution is found.



How to tell if there are more optimal solutions?

- It may be desirable to identify other optimal solutions. Some programs tell you they exist.
- Whenever a problem has more than one optimal BFS, at least one of the non-basic variables will have a coefficient of zero in the final objective function.
- Other solutions can be found by pivoting and using the variable with the zero coefficient in the objective row as the entering variable.

Finding multiple optimal solutions

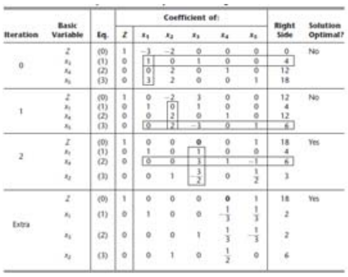


$$\boxed{(x_1, x_2) = w_1(2, 6) + w_2(4, 3),}$$

$$w_1 + w_2 = 1 \quad \text{and} \quad w_1 \ge 0, \quad w_2 \ge 0.$$

- All optimal solutions are a *weighted average of these two optimal CPF solutions*

Complete set of simplex tableaux for the Wyndor Glass Co. problem with $c_2 = 2$



Iteration	Basic Variable	Eq.	Coefficient of:						Right Side	Solution Optimal?
			Z	x_1	x_2	x_3	x_4	x_5		
0	Z	(0)	1	−3	−2	0	0	0	0	No
	x_3	(1)	0	1	0	1	0	0	4	
	x_4	(2)	0	0	2	0	1	0	12	
	x_5	(3)	0	3	2	0	0	1	18	
1	Z	(0)	1	0	−2	3	0	0	12	No
	x_1	(1)	0	1	0	1	0	0	4	
	x_4	(2)	0	0	2	0	1	0	12	
	x_5	(3)	0	0	2	−3	0	1	6	
2	Z	(0)	1	0	0	0	0	1	18	Yes
	x_1	(1)	0	1	0	1	0	0	4	
	x_4	(2)	0	0	0	3	1	−1	6	
	x_2	(3)	0	0	1	$-\frac{3}{2}$	0	$\frac{1}{2}$	3	
Extra	Z	(0)	1	0	0	0	0	1	18	Yes
	x_1	(1)	0	1	0	0	$-\frac{1}{3}$	$\frac{1}{3}$	2	
	x_3	(2)	0	0	0	1	$\frac{1}{3}$	$-\frac{1}{3}$	2	
	x_2	(3)	0	0	1	0	$\frac{1}{2}$	0	6	

$$\boxed{\begin{aligned}&(x_1, x_2, x_3, x_4, x_5) = w_1(2, 6, 2, 0, 0) + w_2(4, 3, 0, 6, 0),\\&w_1 + w_2 = 1, \qquad w_1 \ge 0, \qquad w_2 \ge 0.\end{aligned}}$$

Adapting to Other Model Forms

- Equality Constraints
- Negative Right Hand Sides
- Functional Constraints in ≥ form
- Minimization problems

An Equality Constraint

- An equality constraint is equivalent to two inequality constraints.

$$a_{i1}x_1 + a_{i2}x_2 + \cdots + a_{in}x_n = b_i$$

$$\boxed{\begin{aligned}a_{i1}x_1 + a_{i2}x_2 + \cdots + a_{in}x_n &\le b_i\\a_{i1}x_1 + a_{i2}x_2 + \cdots + a_{in}x_n &\ge b_i.\end{aligned}}$$

- Another way to handle these constraints, with an artificial variable.

Modified Problem

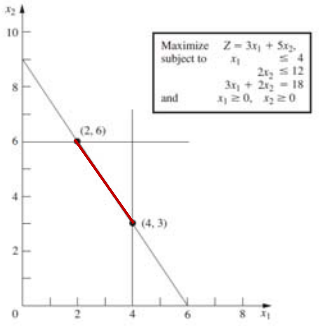
Change the last constraint so that

3X1 + 2X2 = 18

$$
\begin{array}{lcl}
Z - 3X_1 - 5X_2 & & = 0 \\
X_1 + X_3 & & = 4 \\
2X_2 + X_4 & & = 12 \\
3X_1 + 2X_2 & & = 18
\end{array}
$$

There is no obvious BFS because there is no slack in the third constraint.

Modified Problem



Maximize $Z = 3x_1 + 5x_2$,
subject to $x_1 \le 4$
$2x_2 \le 12$
$3x_1 + 2x_2 = 18$
and $x_1 \ge 0, \quad x_2 \ge 0$

How to Obtain an Initial BF Solution?

The procedure is to construct an **artificial problem** that has the same optimal solution as the real problem by making two modifications of the real problem.

1. Apply the **artificial-variable technique by introducing a *nonnegative artificial variable*** (call it $\overline{X_5}$ *) into Eq. (3), just as if it were a slack variable*
2. Assign an *overwhelming penalty to having* $\overline{X_5}$ *> 0 by changing the objective function*

$Z = 3x_1 + 5x_2$ to

$Z = 3x_1 + 5x_2 - M\bar{x}_5$.

Modified Problem

The Real Problem

Maximize $Z = 3x_1 + 5x_2$,

subject to

$x_1 \le 4$

$2x_2 \le 12$

$3x_1 + 2x_2 = 18$

and

$x_1 \ge 0, \quad x_2 \ge 0.$

The Artificial Problem

Define $\bar{x}_5 = 18 - 3x_1 - 2x_2$.

Maximize $Z = 3x_1 + 5x_2 - M\bar{x}_5$,

subject to

$x_1 \le 4$

$2x_2 \le 12$

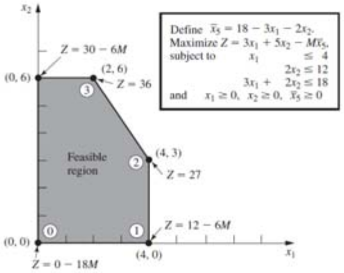
$3x_1 + 2x_2 \le 18$

(so $3x_1 + 2x_2 + \bar{x}_5 = 18$)

and

$x_1 \ge 0, \quad x_2 \ge 0, \quad \bar{x}_5 \ge 0.$

Feasible Region for the Artificial Problem



System of Equations

$$\begin{array}{ll}
(0) & Z - 3x_1 - 5x_2 \qquad + M\bar{x}_5 = 0 \\
(1) & x_1 \qquad + x_3 \qquad = 4 \\
(2) & 2x_2 \qquad + x_4 \qquad = 12 \\
(3) & 3x_1 + 2x_2 \qquad + \bar{x}_5 = 18
\end{array}$$

This system is not yet in proper form from Gaussian elimination because a basic variable X_5 *has a* nonzero coefficient in Eq. (0).

Recall that all basic variables must be algebraically eliminated from Eq. (0) before the simplex method can either apply the optimality test or find the entering basic variable.

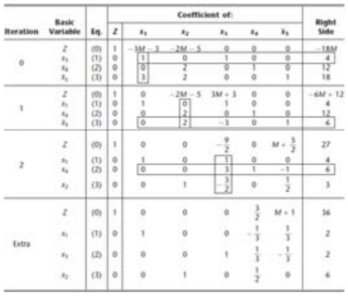
System of Equations

$$\begin{array}{ll}
(0) & Z - 3x_1 - 5x_2 \qquad + M\bar{x}_5 = 0 \\
(1) & x_1 \qquad + x_3 \qquad = 4 \\
(2) & 2x_2 \qquad + x_4 \qquad = 12 \\
(3) & 3x_1 + 2x_2 \qquad + \bar{x}_5 = 18
\end{array}$$

$$\begin{array}{rl}
 & Z - 3x_1 - 5x_2 + M\bar{x}_5 = 0 \\
 & -M(3x_1 + 2x_2 + \bar{x}_5 = 18) \\
\hline
\text{New (0)} & Z - (3M + 3)x_1 - (2M + 5)x_2 = -18M.
\end{array}$$

The quantities involving *M never appear in the system of equations except for Eq.* (0), so they need to be taken into account only in the optimality test and when an entering basic variable is determined.

Complete set of simplex tableaux



Iteration	Basic Variable	Eq.	Z	x_1	x_2	x_3	x_4	$\bar{x}_5$	Right Side
0	Z	(0)	1	$-3M-3$	$-2M-5$	0	0	0	$-18M$
	x_3	(1)	0	1	0	1	0	0	4
	x_4	(2)	0	0	2	0	1	0	12
	$\bar{x}_5$	(3)	0	3	2	0	0	1	18
1	Z	(0)	1	0	$-2M-5$	$3M+3$	0	0	$-6M+12$
	x_1	(1)	0	1	0	1	0	0	4
	x_4	(2)	0	0	2	0	1	0	12
	$\bar{x}_5$	(3)	0	0	2	-3	0	1	6
2	Z	(0)	1	0	0	$-\frac{9}{2}$	0	$M+\frac{5}{2}$	27
	x_1	(1)	0	1	0	1	0	0	4
	x_4	(2)	0	0	0	3	1	-1	6
	x_2	(3)	0	0	1	$-\frac{3}{2}$	0	$\frac{1}{2}$	3
Extra	Z	(0)	1	0	0	0	$\frac{3}{2}$	$M+1$	36
	x_1	(1)	0	1	0	0	$-\frac{1}{3}$	$\frac{1}{3}$	2
	x_3	(2)	0	0	0	1	$\frac{1}{3}$	$-\frac{1}{3}$	2
	x_2	(3)	0	0	1	0	$\frac{1}{2}$	0	6

Negative RHS

- To eliminate the negative value in the RHS, multiply through by -1. This operation reverses the sign of the inequality.

- In doing simplex by hand, all RHS variables must be positive. Not required for computer algorithms.

$$\boxed{x_1 - x_2 \leq -1}$$

$$\boxed{-x_1 + x_2 \geq 1}$$

Functional Constraints in ≥ Form

- We introduce both a surplus variable and an artificial variable so that we can have a BFS.

Radiation Therapy Example

Minimize $Z = 0.4x_1 + 0.5x_2$,

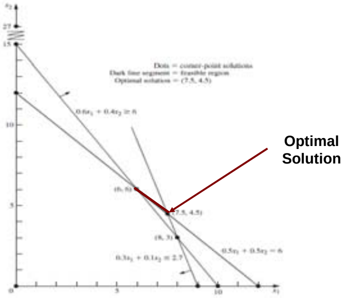
subject to

$$0.3x_1 + 0.1x_2 \leq 2.7$$
$$0.5x_1 + 0.5x_2 = 6$$
$$\boxed{0.6x_1 + 0.4x_2 \geq 6}$$

and

$$x_1 \geq 0, \quad x_2 \geq 0.$$

Mary’s Radiation Therapy Example



Dealing with the Third Constraint

$$\begin{aligned}
&0.6x_1 + 0.4x_2 && \geq 6 && \\
&0.6x_1 + 0.4x_2 - x_5 && = 6 && (x_5 \geq 0) \\
&0.6x_1 + 0.4x_2 - x_5 + \bar{x}_6 && = 6 && (x_5 \geq 0, \bar{x}_6 \geq 0).
\end{aligned}$$

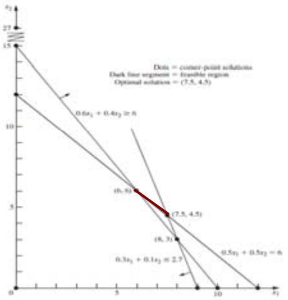
Mary’s Radiation Therapy Example

Minimize	$Z = 0.4x_1 + 0.5x_2 \textcircled{+} M\bar{x}_4 \textcircled{+} M\bar{x}_6,$				
subject to	$0.3x_1 + 0.1x_2 + x_3 = 2.7$				
	$0.5x_1 + 0.5x_2 + \bar{x}_4 = 6$				
	$0.6x_1 + 0.4x_2 - x_5 + \bar{x}_6 = 6$				
and $x_1 \geq 0,$	$x_2 \geq 0,$	$x_3 \geq 0,$	$\bar{x}_4 \geq 0,$	$x_5 \geq 0,$	$\bar{x}_6 \geq 0.$

Introducing artificial variables enlarges the feasible region

Constraints on (x_1, x_2) *for the Real Problem*	*Constraints on* (x_1, x_2) *for the Artificial Problem*
$0.3x_1 + 0.1x_2 \leq 2.7$	$0.3x_1 + 0.1x_2 \leq 2.7$
$0.5x_1 + 0.5x_2 = 6$	$0.5x_1 + 0.5x_2 \leq 6$ $\quad$ ($=$ holds when $\bar{x}_4 = 0$)
$0.6x_1 + 0.4x_2 \geq 6$	No such constraint (except when $\bar{x}_6 = 0$)
$x_1 \geq 0, \quad x_2 \geq 0$	$x_1 \geq 0, \quad x_2 \geq 0$

Mary’s Radiation Therapy Example



How to deal with the minimization problem?

- One straightforward way of minimizing *Z with the simplex method is to exchange the* roles of the positive and negative coefficients in row 0 for both the optimality test and step 1 of an iteration.

- OR,

Minimizing $\quad Z = \sum_{j=1}^{n} c_j x_j$

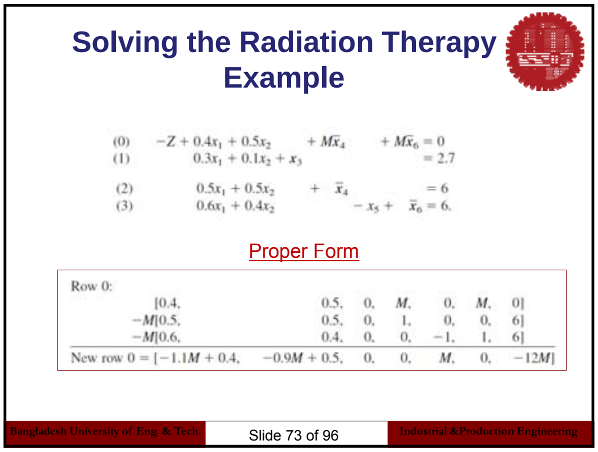
is equivalent to

maximizing $\quad -Z = \sum_{j=1}^{n} (-c_j) x_j;$

Mary’s Radiation Therapy Example

Minimize $\quad Z = 0.4x_1 + 0.5x_2$
Maximize $\quad -Z = -0.4x_1 - 0.5x_2.$

Minimize $\quad Z = 0.4x_1 + 0.5x_2 + M\bar{x}_4 + M\bar{x}_6$
Maximize $\quad -Z = -0.4x_1 - 0.5x_2 - M\bar{x}_4 - M\bar{x}_6.$



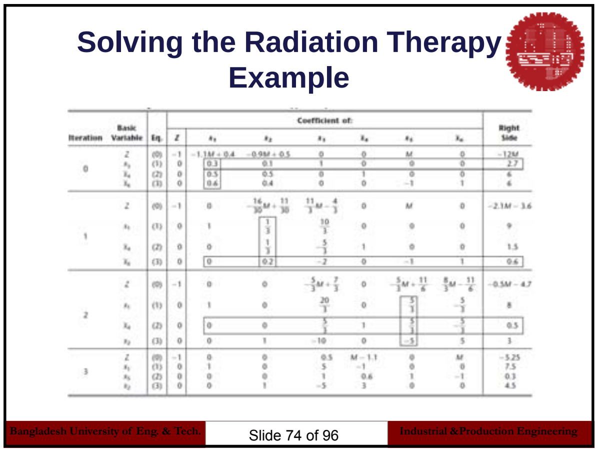
Solving the Radiation Therapy Example

$$
\begin{array}{ll}
(0) & -Z + 0.4x_1 + 0.5x_2 \quad + M\bar{x}_4 \quad + M\bar{x}_6 = 0 \\
(1) & 0.3x_1 + 0.1x_2 + x_3 \qquad = 2.7 \\
(2) & 0.5x_1 + 0.5x_2 \quad + \quad \bar{x}_4 \qquad = 6 \\
(3) & 0.6x_1 + 0.4x_2 \qquad - x_5 + \quad \bar{x}_6 = 6.
\end{array}
$$

Proper Form

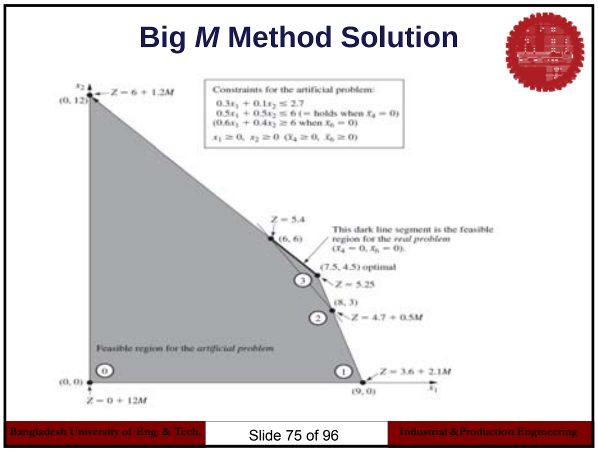
Row 0:

$$
\begin{array}{rrrrrrr}
[0.4, & 0.5, & 0, & M, & 0, & M, & 0] \\
-M[0.5, & 0.5, & 0, & 1, & 0, & 0, & 6] \\
-M[0.6, & 0.4, & 0, & 0, & -1, & 1, & 6] \\
\hline
\text{New row } 0 = [-1.1M + 0.4, & -0.9M + 0.5, & 0, & 0, & M, & 0, & -12M]
\end{array}
$$

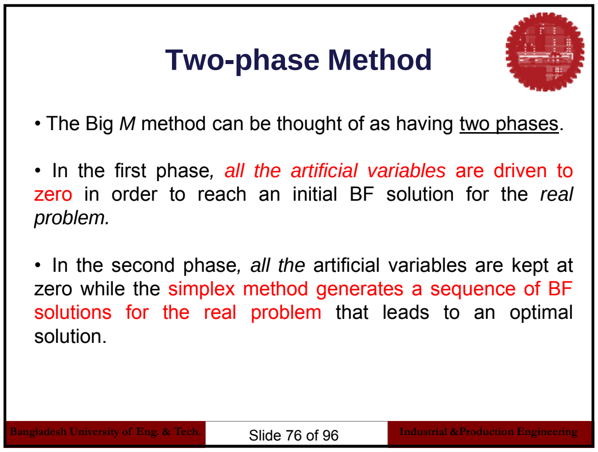


Solving the Radiation Therapy Example

Iteration	Basic Variable	Eq.	Coefficient of: Z	x_1	x_2	x_3	$\bar{x}_4$	x_5	$\bar{x}_6$	Right Side
0	Z	(0)	−1	$-1.1M+0.4$	$-0.9M+0.5$	0	0	M	0	$-12M$
	x_3	(1)	0	0.3	0.1	1	0	0	0	2.7
	$\bar{x}_4$	(2)	0	0.5	0.5	0	1	0	0	6
	$\bar{x}_6$	(3)	0	0.6	0.4	0	0	−1	1	6
1	Z	(0)	−1	0	$-\frac{16}{30}M+\frac{11}{30}$	$\frac{11}{3}M-\frac{4}{3}$	0	M	0	$-2.1M-3.6$
	x_1	(1)	0	1	$\frac{1}{3}$	$\frac{10}{3}$	0	0	0	9
	$\bar{x}_4$	(2)	0	0	$\frac{1}{3}$	$-\frac{5}{3}$	1	0	0	1.5
	$\bar{x}_6$	(3)	0	0	0.2	−2	0	−1	1	0.6
2	Z	(0)	−1	0	0	$-\frac{5}{3}M+\frac{7}{3}$	0	$-\frac{5}{3}M+\frac{11}{6}$	$\frac{8}{3}M-\frac{11}{6}$	$-0.5M-4.7$
	x_1	(1)	0	1	0	$\frac{20}{3}$	0	$\frac{5}{3}$	$-\frac{5}{3}$	8
	$\bar{x}_4$	(2)	0	0	0	$\frac{5}{3}$	1	$\frac{5}{3}$	$-\frac{5}{3}$	0.5
	x_2	(3)	0	0	1	−10	0	−5	5	3
3	Z	(0)	−1	0	0	0.5	$M-1.1$	0	M	−5.25
	x_1	(1)	0	1	0	5	−1	0	0	7.5
	x_5	(2)	0	0	0	1	0.6	1	−1	0.3
	x_2	(3)	0	0	1	−5	3	0	0	4.5



Big *M* Method Solution



Two-phase Method

- The Big *M* method can be thought of as having two phases.
- In the first phase, *all the artificial variables are driven to zero* in order to reach an initial BF solution for the *real problem.*
- In the second phase, *all the* artificial variables are kept at zero while the simplex method generates a sequence of BF solutions for the real problem that leads to an optimal solution.

Two-phase Method

Real problem: Minimize $Z = 0.4x_1 + 0.5x_2$.

Big M method: Minimize $Z = 0.4x_1 + 0.5x_2 + M\bar{x}_4 + M\bar{x}_6$.

Two-phase method:

Phase 1: Minimize $Z = \bar{x}_4 + \bar{x}_6$ (until $\bar{x}_4 = 0, \bar{x}_6 = 0$).

Phase 2: Minimize $Z = 0.4x_1 + 0.5x_2$ (with $\bar{x}_4 = 0, \bar{x}_6 = 0$).

Summary of the Two-Phase Method

- Initialization: Revise the constraints of the original problem by introducing artificial variables as needed to obtain an obvious initial BF solution for the artificial problem.
- Phase 1: The objective for this phase is to find a BF solution for the real problem.

  (Minimize $Z = \Sigma$ *artificial variables, subject to revised constraints*).
- Phase 2: The objective for this phase is to find an optimal solution for the real problem.

Two-phase Method: Radiation Therapy Example

Phase 1 Problem (Radiation Therapy Example):

Minimize $Z = \bar{x}_4 + \bar{x}_6$,

subject to

$$\begin{aligned} 0.3x_1 + 0.1x_2 + x_3 &= 2.7 \\ 0.5x_1 + 0.5x_2 + \bar{x}_4 &= 6 \\ 0.6x_1 + 0.4x_2 - x_5 + \bar{x}_6 &= 6 \end{aligned}$$

and

$x_1 \geq 0, \quad x_2 \geq 0, \quad x_3 \geq 0, \quad \bar{x}_4 \geq 0, \quad x_5 \geq 0, \quad \bar{x}_6 \geq 0.$

Phase 2 Problem (Radiation Therapy Example):

Minimize $Z = 0.4x_1 + 0.5x_2$,

subject to

$$\begin{aligned} 0.3x_1 + 0.1x_2 + x_3 &= 2.7 \\ 0.5x_1 + 0.5x_2 &= 6 \\ 0.6x_1 + 0.4x_2 - x_5 &= 6 \end{aligned}$$

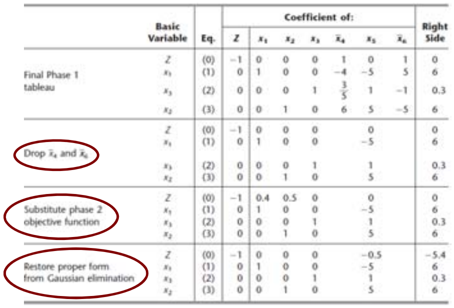
and

$x_1 \geq 0, \quad x_2 \geq 0, \quad x_3 \geq 0, \quad x_5 \geq 0.$

Phase 1 of two-phase method for radiation therapy example

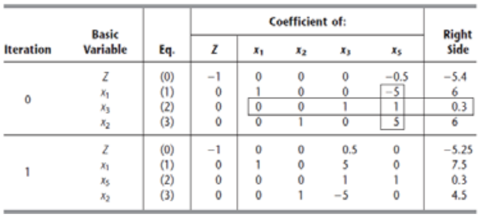
Iteration	Basic Variable	Eq.	Coefficient of: Z	x_1	x_2	x_3	$\bar{x}_4$	x_5	$\bar{x}_6$	Right Side
0	Z	(0)	−1	−1.1	−0.9	0	0	1	0	−12
	x_3	(1)	0	0.3	0.1	1	0	0	0	2.7
	$\bar{x}_4$	(2)	0	0.5	0.5	0	1	0	0	6
	$\bar{x}_6$	(3)	0	0.6	0.4	0	0	−1	1	6
1	Z	(0)	−1	0	$-\frac{16}{30}$	$\frac{11}{3}$	0	1	0	−2.1
	x_1	(1)	0	1	$\frac{1}{3}$	$\frac{10}{3}$	0	0	0	9
	$\bar{x}_4$	(2)	0	0	$\frac{1}{3}$	$-\frac{5}{3}$	1	0	0	1.5
	$\bar{x}_6$	(3)	0	0	0.2	−2	0	−1	1	0.6
2	Z	(0)	−1	0	0	$-\frac{5}{3}$	0	$-\frac{5}{3}$	$\frac{8}{3}$	−0.5
	x_1	(1)	0	1	0	$\frac{20}{3}$	0	$\frac{5}{3}$	$-\frac{5}{3}$	8
	$\bar{x}_4$	(2)	0	0	0	$\frac{5}{3}$	1	$\frac{5}{3}$	$-\frac{5}{3}$	0.5
	x_2	(3)	0	0	1	−10	0	−5	5	3
3	Z	(0)	−1	0	0	0	1	0	1	0
	x_1	(1)	0	1	0	0	−4	−5	5	6
	x_3	(2)	0	0	0	1	$\frac{3}{5}$	1	−1	0.3
	x_2	(3)	0	0	1	0	6	5	−5	6

Preparing to begin phase 2 for radiation therapy example



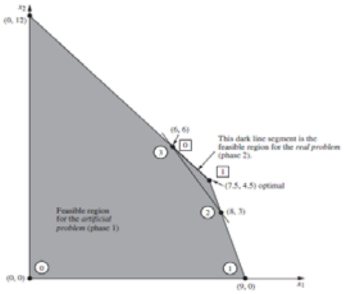
	Basic Variable	Eq.	Coefficient of: Z	x_1	x_2	x_3	$\bar{x}_4$	x_5	$\bar{x}_6$	Right Side
Final Phase 1 tableau	Z	(0)	−1	0	0	0	1	0	1	0
	x_1	(1)	0	1	0	0	−4	−5	5	6
	x_3	(2)	0	0	0	1	$\frac{3}{5}$	1	−1	0.3
	x_2	(3)	0	0	1	0	6	5	−5	6
Drop $\bar{x}_4$ and $\bar{x}_6$	Z	(0)	−1	0	0	0		0		0
	x_1	(1)	0	1	0	0		−5		6
	x_3	(2)	0	0	0	1		1		0.3
	x_2	(3)	0	0	1	0		5		6
Substitute phase 2 objective function	Z	(0)	−1	0.4	0.5	0		0		0
	x_1	(1)	0	1	0	0		−5		6
	x_3	(2)	0	0	0	1		1		0.3
	x_2	(3)	0	0	1	0		5		6
Restore proper form from Gaussian elimination	Z	(0)	−1	0	0	0		−0.5		−5.4
	x_1	(1)	0	1	0	0		−5		6
	x_3	(2)	0	0	0	1		1		0.3
	x_2	(3)	0	0	1	0		5		6

Phase 2 of two-phase method for radiation therapy example



Iteration	Basic Variable	Eq.	Coefficient of: Z	x_1	x_2	x_3	x_5	Right Side
0	Z	(0)	−1	0	0	0	−0.5	−5.4
	x_1	(1)	0	1	0	0	−5	6
	x_3	(2)	0	0	0	1	1	0.3
	x_2	(3)	0	0	1	0	5	6
1	Z	(0)	−1	0	0	0.5	0	−5.25
	x_1	(1)	0	1	0	5	0	7.5
	x_5	(2)	0	0	0	1	1	0.3
	x_2	(3)	0	0	1	−5	0	4.5

Two-phase Method: Radiation Therapy Solution



No Feasible Solutions

If the original problem has *no feasible solutions, then either the Big M method or phase* 1 of the two-phase method yields a final solution that has at least one artificial variable *greater than zero. Otherwise, they all equal zero.*

Variables Allowed to Be Negative

- The procedure for determining the *leaving basic variable* requires that all the variables have nonnegativity constraints.

- Any problem containing variables allowed to be negative must be converted to an *equivalent problem involving only nonnegative variables* before the simplex method is applied.

Variables with a Bound on the Negative Values Allowed

$$x_j \ge L_j,$$

$$\boxed{x'_j = x_j - L_j, \qquad \text{so} \qquad x'_j \ge 0.}$$

Variables with No Bound on the Negative Values Allowed

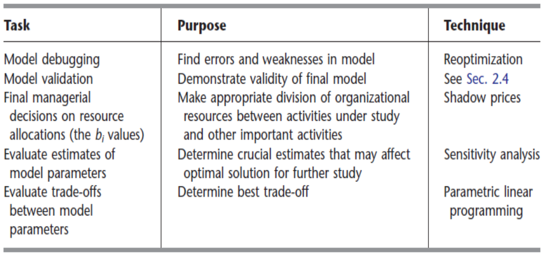
$$\boxed{x_j = x_j^+ - x_j^-, \qquad \text{where } x_j^+ \ge 0, x_j^- \ge 0.}$$

$$\boxed{x_j = x'_j - x'', \qquad \text{where } x'_j \ge 0, x'' \ge 0,}$$

Postoptimality Analysis

- *Postoptimality analysis* — the analysis done *after* an optimal solution is obtained for the initial version of the model.

Postoptimality analysis for LP



Task	Purpose	Technique
Model debugging	Find errors and weaknesses in model	Reoptimization
Model validation	Demonstrate validity of final model	See Sec. 2.4
Final managerial decisions on resource allocations (the b_i values)	Make appropriate division of organizational resources between activities under study and other important activities	Shadow prices
Evaluate estimates of model parameters	Determine crucial estimates that may affect optimal solution for further study	Sensitivity analysis
Evaluate trade-offs between model parameters	Determine best trade-off	Parametric linear programming

Reoptimization

- Reoptimization involves deducing how changes in the model get carried along to the *final* simplex tableau.

- This revised tableau and the optimal solution for the prior model are then used as the *initial tableau and the initial basic solution for solving the new model.*

- *If this solution is feasible for the new model, then* the simplex method is applied in the usual way, starting from this initial BF solution.

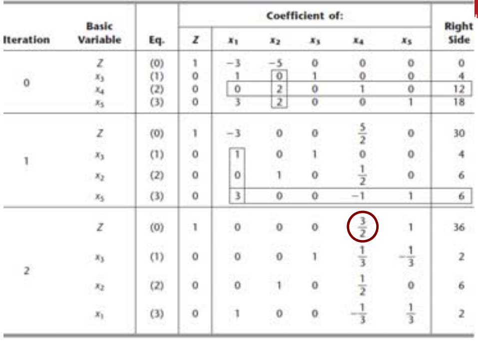
- If the solution is not feasible, a related algorithm called the *dual simplex method* probably can be applied to find the new optimal solution, starting from this initial basic solution.

Shadow Price

- The **shadow price for resource *i* *(denoted by y_i*)* *measures the marginal value of this resource,*** i.e., the rate at which *Z could be increased by (slightly) increasing the amount* of this resource (b_i) being made available.

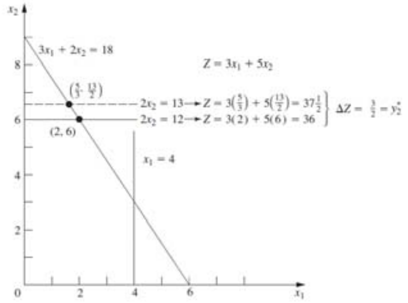
- *The simplex method identifies this shadow price* by y_i^* = *coefficient of the ith slack variable in row 0 of the final simplex tableau.*

Wyndor Problem



Iteration	Basic Variable	Eq.	Coefficient of: Z	x_1	x_2	x_3	x_4	x_5	Right Side
0	Z	(0)	1	−3	−5	0	0	0	0
	x_3	(1)	0	1	0	1	0	0	4
	x_4	(2)	0	0	2	0	1	0	12
	x_5	(3)	0	3	2	0	0	1	18
1	Z	(0)	1	−3	0	0	$\frac{5}{2}$	0	30
	x_3	(1)	0	1	0	1	0	0	4
	x_2	(2)	0	0	1	0	$\frac{1}{2}$	0	6
	x_5	(3)	0	3	0	0	−1	1	6
2	Z	(0)	1	0	0	0	$\frac{3}{2}$	1	36
	x_3	(1)	0	0	0	1	$\frac{1}{3}$	$-\frac{1}{3}$	2
	x_2	(2)	0	0	1	0	$\frac{1}{2}$	0	6
	x_1	(3)	0	1	0	0	$-\frac{1}{3}$	$\frac{1}{3}$	2

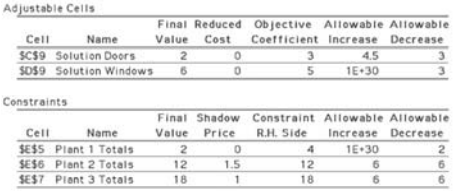
Wyndor Problem



Sensitivity Analysis

- A main purpose of sensitivity analysis is to identify the **sensitive parameters** (i.e., those that cannot be changed without changing the optimal solution).

Allowable range to stay optimal and feasible



Adjustable Cells

Cell	Name	Final Value	Reduced Cost	Objective Coefficient	Allowable Increase	Allowable Decrease
C9	Solution Doors	2	0	3	4.5	3
D9	Solution Windows	6	0	5	1E+30	3

Constraints

Cell	Name	Final Value	Shadow Price	Constraint R.H. Side	Allowable Increase	Allowable Decrease
E5	Plant 1 Totals	2	0	4	1E+30	2
E6	Plant 2 Totals	12	1.5	12	6	6
E7	Plant 3 Totals	18	1	18	6	6

Assignment

- Problems 4.1-1, 4.1-2, 4.1-3, 4.1-9, 4.3-4, 4.5-1, 4.5-2, 4.6-3, 4.6-4, 4.6-6, 4.6-13, 4.6-15, 4.6-18, 4.7-7,