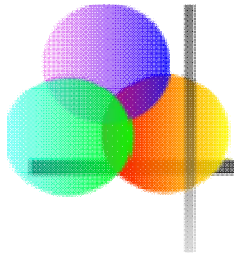
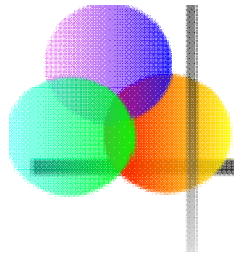


Chapter 5

Some Discrete Probability Distributions

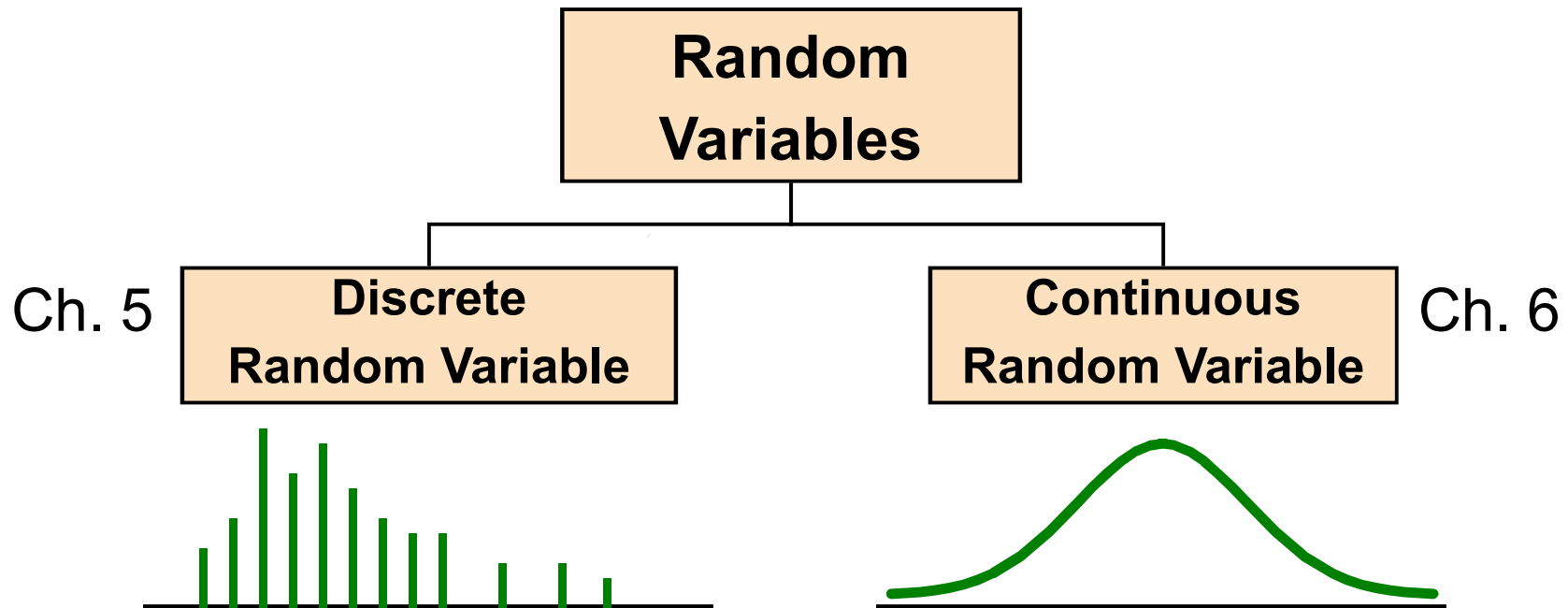


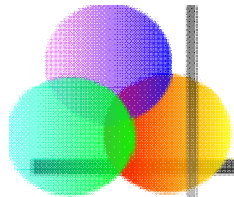


Introduction to Probability Distributions

- **Random Variable**

- Represents a possible numerical value from a random experiment

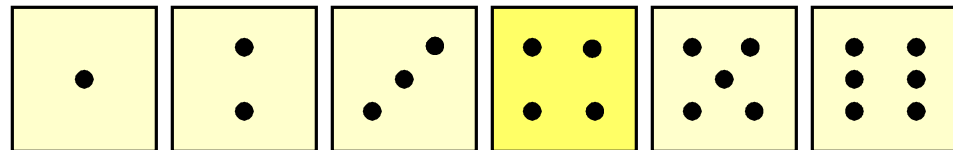




Discrete Random Variables

- Can only take on a countable number of values

Examples:



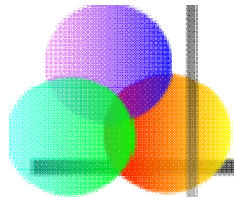
- Roll a die twice**

**Let X be the number of times 4 comes up
(then X could be 0, 1, or 2 times)**

- Toss a coin 5 times.**

**Let X be the number of heads
(then $X = 0, 1, 2, 3, 4, \text{ or } 5$)**



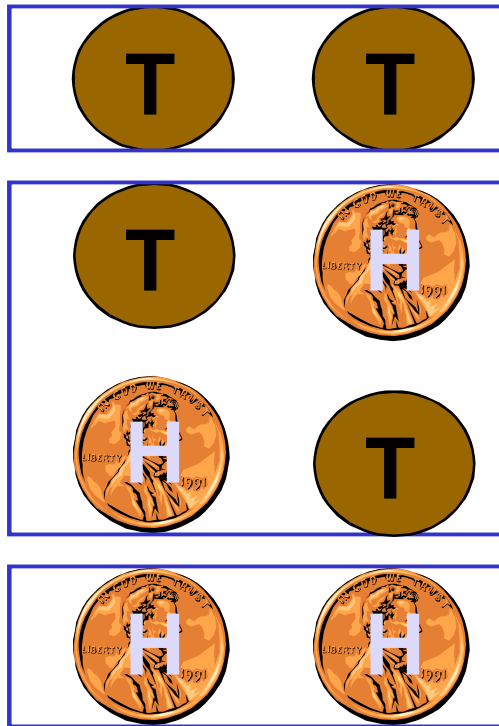


Discrete Probability Distribution

Experiment: Toss 2 Coins. Let $X = \#$ heads.

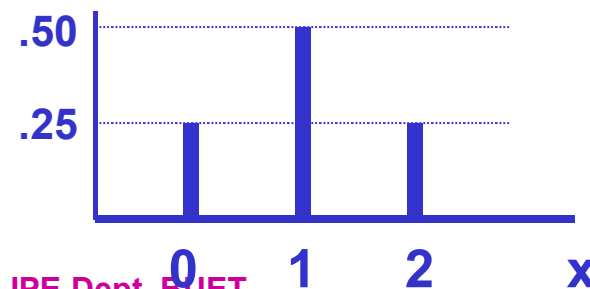
Show $P(x)$, i.e., $P(X = x)$, for all values of x :

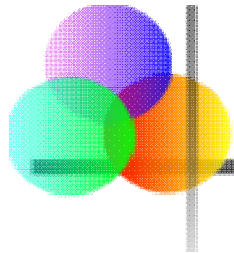
4 possible outcomes



Probability Distribution

<u>x Value</u>	<u>Probability</u>
0	$1/4 = .25$
1	$2/4 = .50$
2	$1/4 = .25$



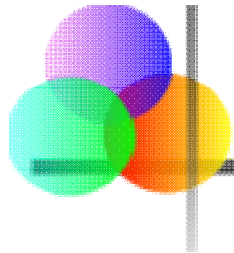


Probability Distribution Required Properties

- $P(x) \geq 0$ for any value of x
- The individual probabilities **sum to 1**;

$$\sum_x P(x) = 1$$

(The notation indicates summation over all possible x values)



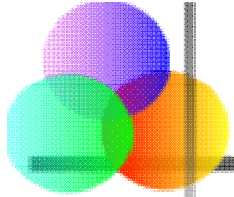
Cumulative Probability Function

- The **cumulative probability function**, denoted $F(x_0)$, shows the probability that X is less than or equal to x_0

$$F(x_0) = P(X \leq x_0)$$

- In other words,

$$F(x_0) = \sum_{x \leq x_0} P(x)$$



Expected Value

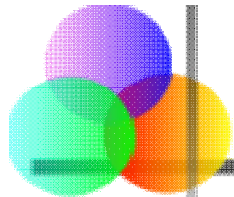
- Expected Value (or mean) of a discrete distribution (Weighted Average)

$$\mu = E(x) = \sum_x xP(x)$$

- Example:** Toss 2 coins,
 $x = \#$ of heads,
compute expected value of x :

$$E(x) = (0 \times .25) + (1 \times .50) + (2 \times .25) \\ = 1.0$$

x	P(x)
0	.25
1	.50
2	.25



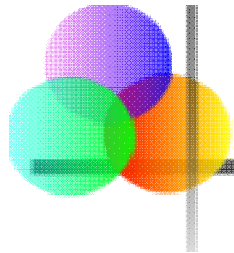
Variance and Standard Deviation

- **Variance** of a discrete random variable X

$$\sigma^2 = E(X - \mu)^2 = \sum_x (x - \mu)^2 P(x)$$

- **Standard Deviation** of a discrete random variable X

$$\sigma = \sqrt{\sigma^2} = \sqrt{\sum_x (x - \mu)^2 P(x)}$$



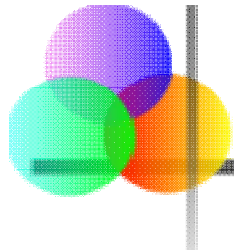
Standard Deviation Example

- **Example:** Toss 2 coins, $X = \#$ heads, compute standard deviation (recall $E(x) = 1$)

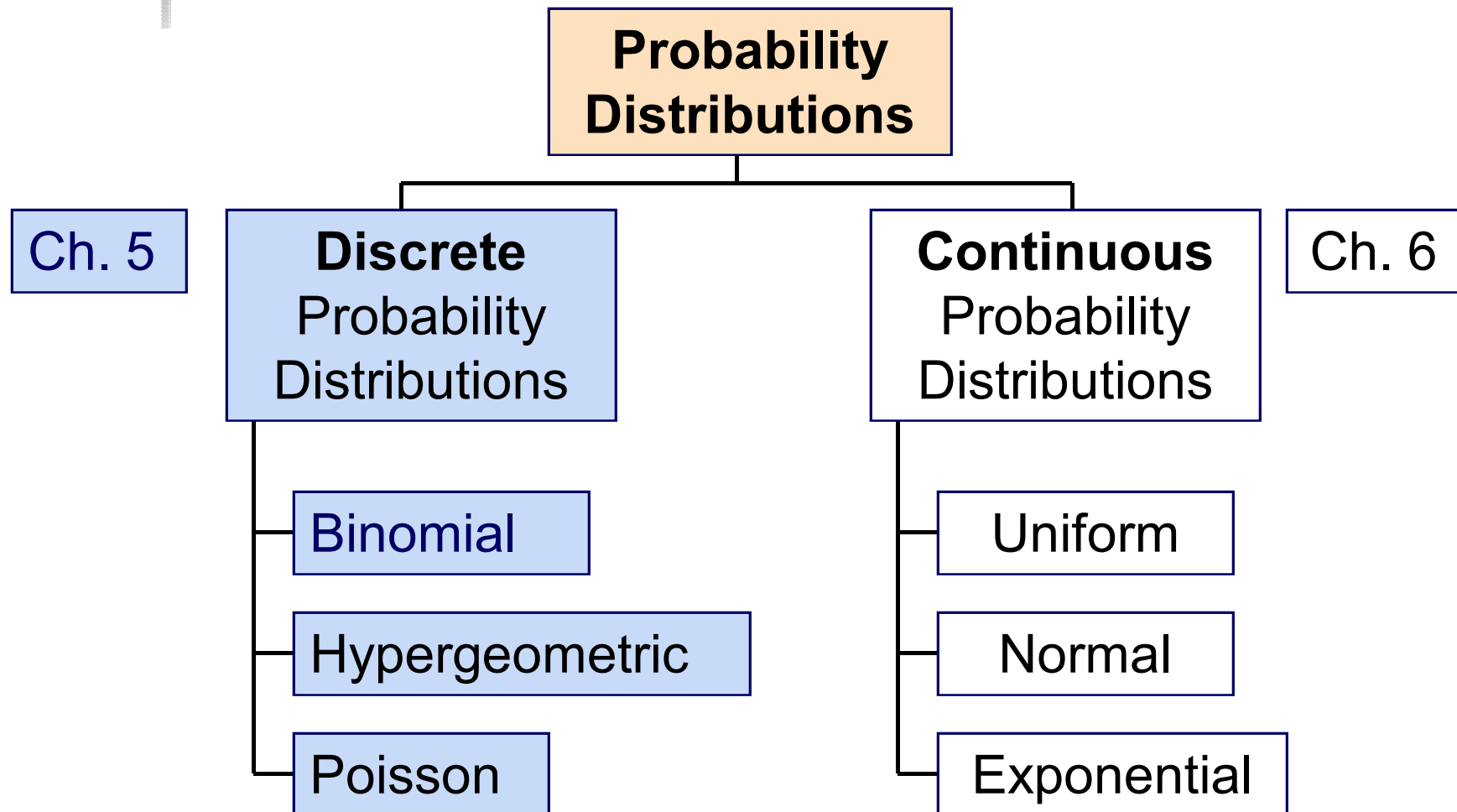
$$\sigma = \sqrt{\sum_x (x - \mu)^2 P(x)}$$

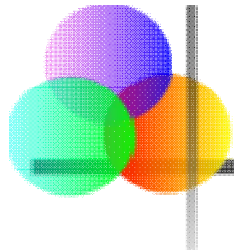
$$\sigma = \sqrt{(0 - 1)^2 (.25) + (1 - 1)^2 (.50) + (2 - 1)^2 (.25)} = \sqrt{.50} = .707$$

Possible number of heads
= 0, 1, or 2

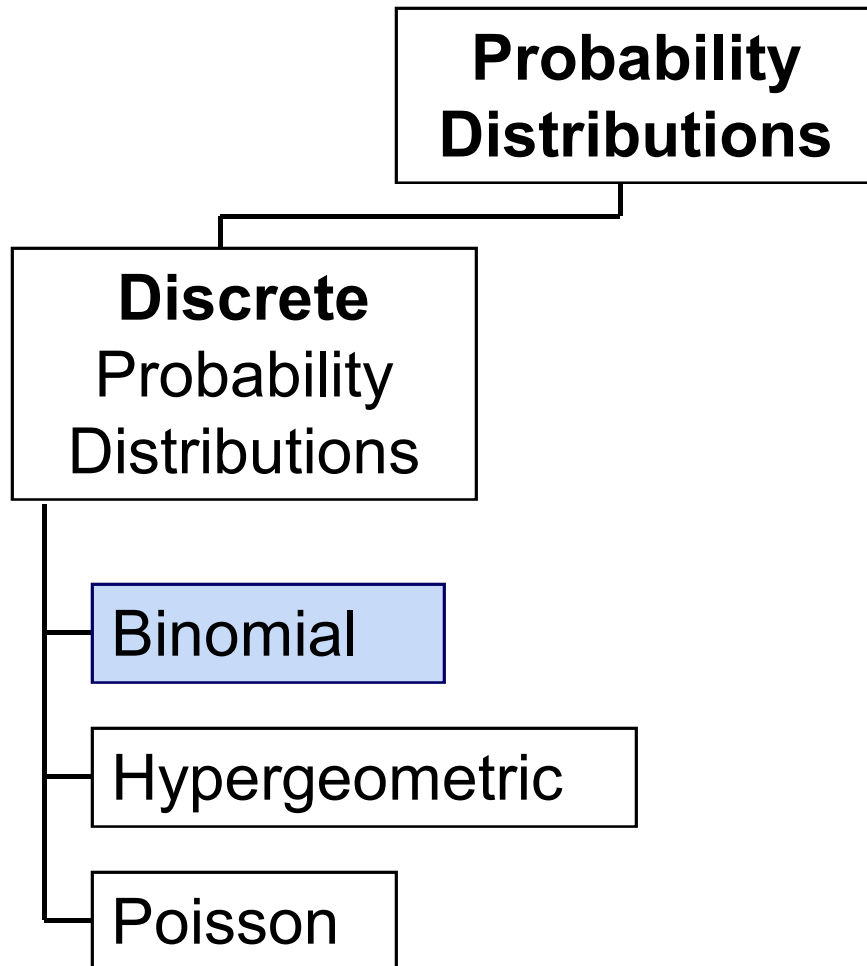


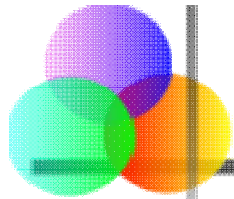
Probability Distributions





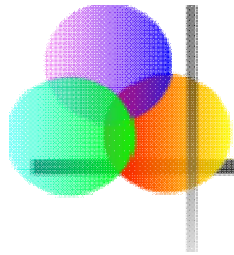
The Binomial Distribution





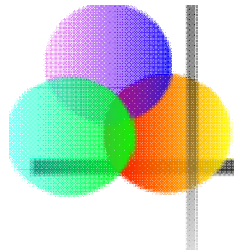
Binomial Probability Distribution

- A fixed number of observations, n
 - e.g., 15 tosses of a coin; ten light bulbs taken from a warehouse
- Two mutually exclusive and collectively exhaustive categories
 - e.g., head or tail in each toss of a coin; defective or not defective light bulb
 - Generally called “success” and “failure”
 - Probability of success is P , probability of failure is $1 - P$
- Constant probability for each observation
 - e.g., Probability of getting a tail is the same each time we toss the coin
- Observations are independent
 - The outcome of one observation does not affect the outcome of the other



Possible Binomial Distribution Settings

- A manufacturing plant labels items as either defective or acceptable
- A firm bidding for contracts will either get a contract or not
- A marketing research firm receives survey responses of “yes I will buy” or “no I will not”
- New job applicants either accept the offer or reject it



Binomial Distribution Formula

$$P(x) = \frac{n!}{x! (n - x)!} P^x (1 - P)^{n - x}$$

$P(x)$ = probability of x successes in n trials,
with probability of success P on each trial

x = number of 'successes' in sample,
($x = 0, 1, 2, \dots, n$)

n = sample size (number of trials
or observations)

P = probability of "success"

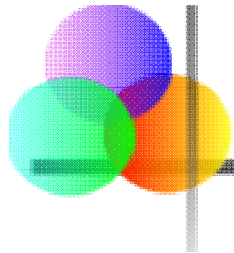
Example: Flip a coin four
times, let x = # heads:

$$n = 4$$

$$P = 0.5$$

$$1 - P = (1 - 0.5) = 0.5$$

$$x = 0, 1, 2, 3, 4$$

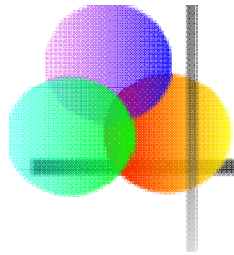


Example: Calculating a Binomial Probability

What is the probability of one success in five observations if the probability of success is 0.1?

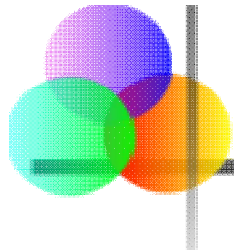
$$x = 1, n = 5, \text{ and } P = 0.1$$

$$\begin{aligned}P(x = 1) &= \frac{n!}{x!(n-x)!} P^x (1-P)^{n-x} \\&= \frac{5!}{1!(5-1)!} (0.1)^1 (1-0.1)^{5-1} \\&= (5)(0.1)(0.9)^4 \\&= .32805\end{aligned}$$



Example: Calculating a Binomial Probability

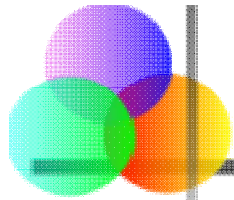
Example 5.5:1: The probability that a patient recovers from a rare blood disease is 0.4. If 15 people are known to have contracted this disease, what is the probability that (a) at least 10 survive, (b) from 3 to 8 survive, and (c) exactly 5 survive?



Example: Calculating a Binomial Probability

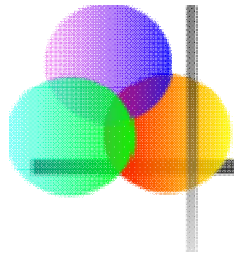
Table A.1 Binomial Probability Sums $\sum_{x=0}^r b(x; n, p)$

<i>n</i>	<i>r</i>	<i>p</i>									
		0.10	0.20	0.25	0.30	0.40	0.50	0.60	0.70	0.80	0.90
1	0	0.9000	0.8000	0.7500	0.7000	0.6000	0.5000	0.4000	0.3000	0.2000	0.1000
	1	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
2	0	0.8100	0.6400	0.5625	0.4900	0.3600	0.2500	0.1600	0.0900	0.0400	0.0100
	1	0.9900	0.9600	0.9375	0.9100	0.8400	0.7500	0.6400	0.5100	0.3600	0.1900
	2	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
3	0	0.7290	0.5120	0.4219	0.3430	0.2160	0.1250	0.0640	0.0270	0.0080	0.0010
	1	0.9720	0.8960	0.8438	0.7840	0.6480	0.5000	0.3520	0.2160	0.1040	0.0280
	2	0.9990	0.9920	0.9844	0.9730	0.9360	0.8750	0.7840	0.6570	0.4880	0.2710
	3	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
4	0	0.6561	0.4096	0.3164	0.2401	0.1296	0.0625	0.0256	0.0081	0.0016	0.0001
	1	0.9477	0.8192	0.7383	0.6517	0.4752	0.3125	0.1792	0.0837	0.0272	0.0037
	2	0.9963	0.9728	0.9492	0.9163	0.8208	0.6875	0.5248	0.3483	0.1808	0.0523
	3	0.9999	0.9984	0.9961	0.9919	0.9744	0.9375	0.8704	0.7599	0.5904	0.3439
	4	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000



Example: Calculating a Binomial Probability

		$\bar{x}=0$									
		<i>P</i>									
<i>n</i>	<i>r</i>	0.10	0.20	0.25	0.30	0.40	0.50	0.60	0.70	0.80	0.90
15	0	0.2059	0.0352	0.0134	0.0047	0.0005	0.0000				
	1	0.5490	0.1671	0.0802	0.0353	0.0052	0.0005	0.0000			
	2	0.8159	0.3980	0.2361	0.1268	0.0271	0.0037	0.0003	0.0000		
	3	0.9444	0.6482	0.4613	0.2969	0.0905	0.0176	0.0019	0.0001		
	4	0.9873	0.8358	0.6865	0.5155	0.2173	0.0592	0.0093	0.0007	0.0000	
	5	0.9978	0.9389	0.8516	0.7216	0.4032	0.1509	0.0338	0.0037	0.0001	
	6	0.9997	0.9819	0.9434	0.8689	0.6098	0.3036	0.0950	0.0152	0.0008	
	7	1.0000	0.9958	0.9827	0.9500	0.7869	0.5000	0.2131	0.0500	0.0042	0.0000
	8		0.9992	0.9958	0.9848	0.9050	0.6964	0.3902	0.1311	0.0181	0.0003
	9		0.9999	0.9992	0.9963	0.9662	0.8491	0.5968	0.2784	0.0611	0.0022
	10		1.0000	0.9999	0.9993	0.9907	0.9408	0.7827	0.4845	0.1642	0.0127
	11			1.0000	0.9999	0.9981	0.9824	0.9095	0.7031	0.3518	0.0556
	12				1.0000	0.9997	0.9963	0.9729	0.8732	0.6020	0.1841
	13					1.0000	0.9995	0.9948	0.9647	0.8329	0.4510
	14						1.0000	0.9995	0.9953	0.9648	0.7941
	15							1.0000	1.0000	1.0000	1.0000



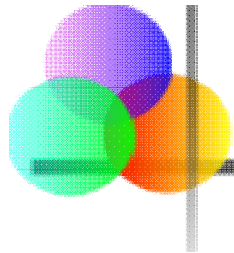
Example: Calculating a Binomial Probability

Solution: Let X be the number of people that survive.

$$(a) \quad P(X \geq 10) = 1 - P(X < 10) = \sum_{x=0}^9 b(x; 15, 0.4) = 1 - 0.9662 = 0.0338$$

$$(b) \quad P(3 \leq X \leq 8) = \sum_{x=3}^8 b(x; 15, 0.4) = \sum_{x=0}^8 b(x; 15, 0.4) - \sum_{x=0}^2 b(x; 15, 0.4) = 0.9 - 0.02 = 0.87$$

$$(c) \quad P(X=5) = b(5; 15, 0.4) = \sum_{x=0}^5 b(x; 15, 0.4) - \sum_{x=0}^4 b(x; 15, 0.4) = 0.4 - 0.22 = 0.18$$



Binomial Distribution

Mean and Variance

- Mean $\mu = E(x) = nP$

- Variance and Standard Deviation

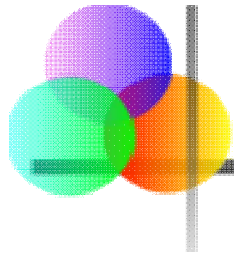
$$\sigma^2 = nP(1-P)$$

$$\sigma = \sqrt{nP(1-P)}$$

Where n = sample size

P = probability of success

$(1 - P)$ = probability of failure

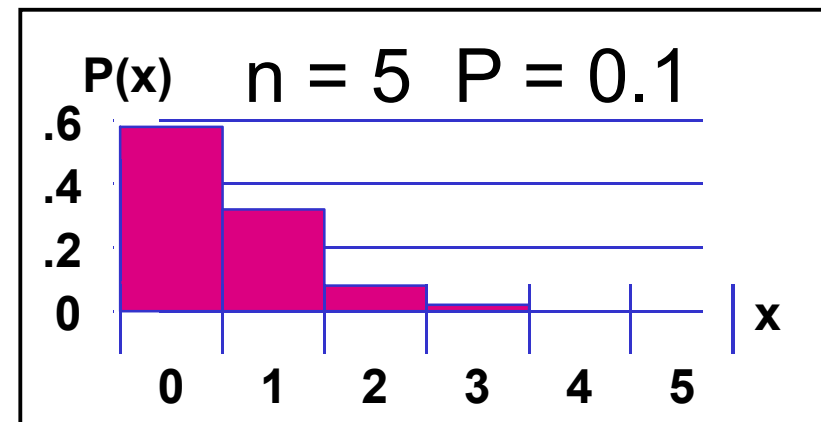


Binomial Characteristics

Examples

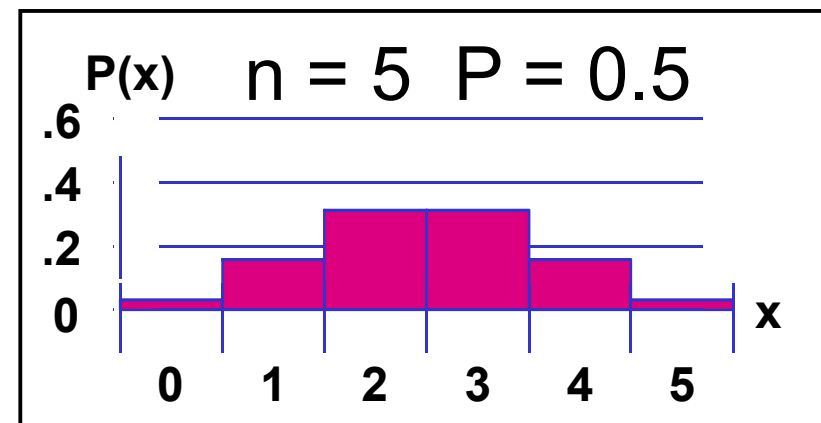
$$\mu = nP = (5)(0.1) = 0.5$$

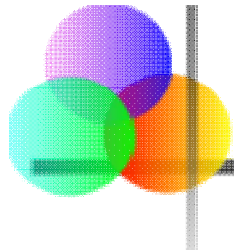
$$\sigma = \sqrt{nP(1-P)} = \sqrt{(5)(0.1)(1-0.1)} = 0.6708$$



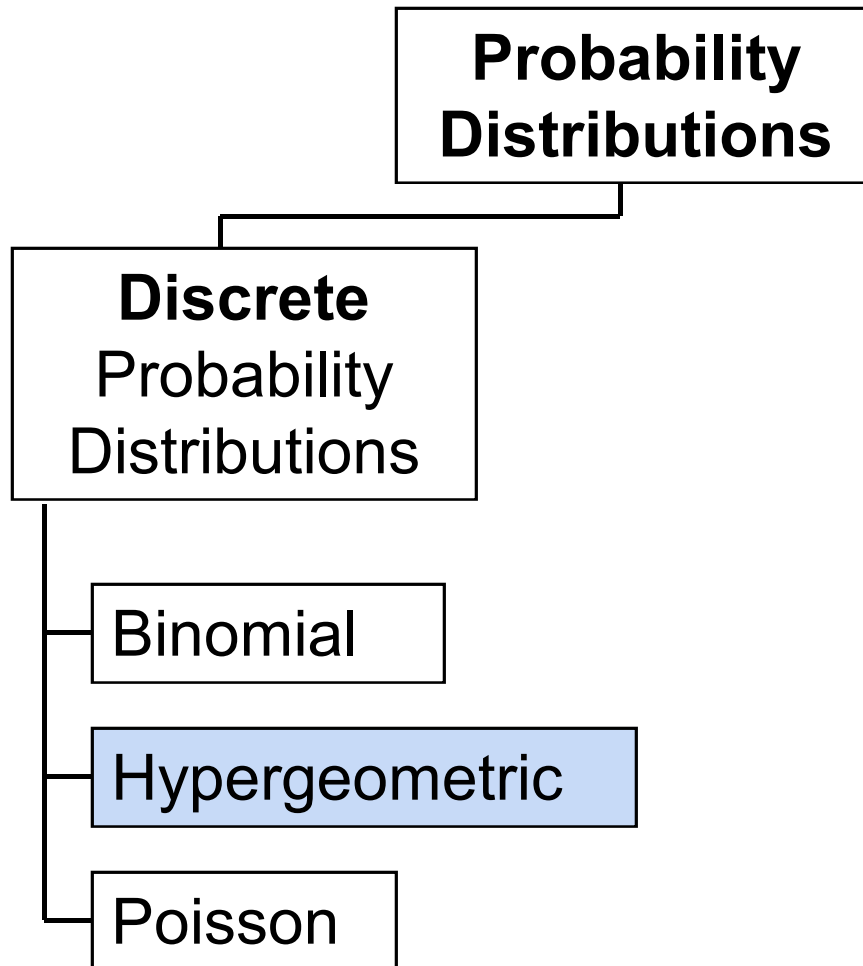
$$\mu = nP = (5)(0.5) = 2.5$$

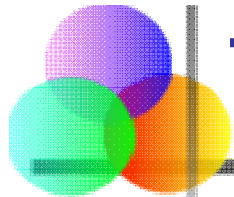
$$\sigma = \sqrt{nP(1-P)} = \sqrt{(5)(0.5)(1-0.5)} = 1.118$$





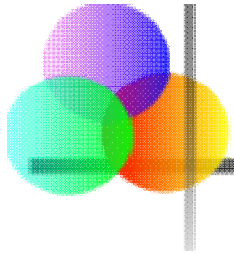
The Hypergeometric Distribution





The Hypergeometric Distribution

- “n” trials in a sample taken from a **finite population** of size N
- Sample taken **without replacement**
- Outcomes of trials are **dependent**
- Concerned with finding the probability of “X” successes in the sample where there are “k” successes in the population



Hypergeometric Distribution Formula

$$P(x) = \frac{C_x^k C_{n-x}^{N-k}}{C_n^N} = \frac{k!}{x!(k-x)!} \times \frac{(N-k)!}{(n-x)!(N-k-n+x)!} \times \frac{N!}{n!(N-n)!}$$

Where

N = population size

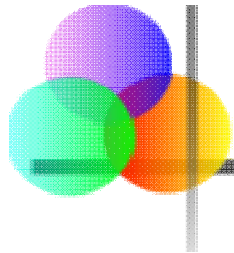
k = number of successes in the population

$N - k$ = number of failures in the population

n = sample size

x = number of successes in the sample

$n - x$ = number of failures in the sample



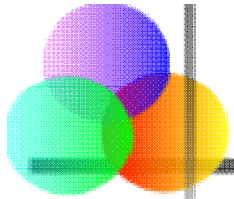
Using the Hypergeometric Distribution

- **Example:** 3 different computers are checked from 10 in the department. 4 of the 10 computers have illegal software loaded. What is the probability that 2 of the 3 selected computers have illegal software loaded?

$$\begin{array}{ll} N = 10 & n = 3 \\ k = 4 & x = 2 \end{array}$$

$$P(x=2) = \frac{C_x^k C_{n-x}^{N-k}}{C_n^N} = \frac{C_2^4 C_1^6}{C_3^{10}} = \frac{(6)(6)}{120} = 0.3$$

The probability that 2 of the 3 selected computers have illegal software loaded is 0.30, or 30%.

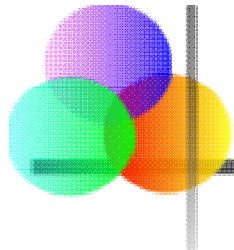


Example 5.12: Lots of 40 components each are deemed unacceptable if they contain 3 or more defectives. The procedure for sampling a lot is to select 5 components at random and to reject the lot if a defective is found. What is the probability that exactly 1 defective is found in the sample if there are 3 defectives in the entire lot?

Solution: Using the hypergeometric distribution with $n = 5$, $N=40$, $k=3$ and $x=1$, we find the probability of obtaining 1 defective to be

$$h(1; 40,5,3) = \frac{\binom{3}{1} \binom{37}{4}}{\binom{40}{5}} = 0.3011$$

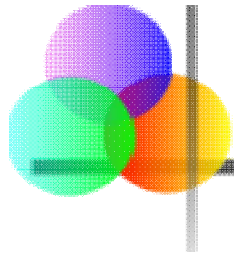
Once again, this plan is not desirable since it detects a bad lot (3 defectives) only about 30% of the time.



Theorem 5.3:

The mean and variance of the hypergeometric distribution $h(x; N, n, k)$ are

$$\mu = \frac{nk}{N} \text{ and } \sigma^2 = \frac{N-n}{N-1} \cdot n \cdot \frac{k}{N} \left(1 - \frac{k}{N}\right).$$

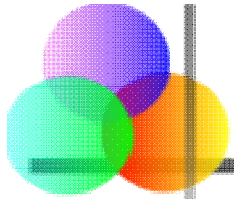


Relationship to the Binomial Distribution

If n is small compared to N , the nature of the N items changes very little in each draw. So a binomial distribution can be used to approximate the hypergeometric distribution when n is small compared to N . As a rule of thumb, the approximation is good when **$n/N \leq 0.05$** .

So, the quantity **k/N** plays the role of the binomial parameter **p** . So, the binomial distribution may be viewed as a large-population version of the hypergeometric distribution. The

$$\mu = np = \frac{nk}{N} \text{ and } \sigma^2 = npq = n \cdot \frac{k}{N} \left(1 - \frac{k}{N} \right).$$

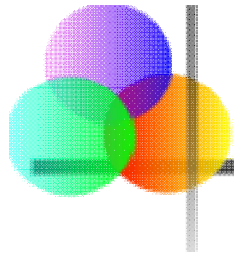


Example 5.15: A manufacturer of automobile tires reports that among a shipment of 5000 sent to a local distributor, 1000 are slightly blemished. If one purchases 10 of these tires at random from the distributor, what is the probability that exactly 3 are blemished?

Solution: As $N = 5000$ is large relative to the sample size $n = 10$, we shall approximate the desired probability by using the binomial distribution. The probability of obtaining a blemished tire is 0.2. So, the probability of obtaining exactly 3 blemished tires is

$$h(3; 5000, 10, 1000) \approx b(3; 10, 0.2) = \sum_{x=0}^3 b(x; 10, 0.2) - \sum_{x=0}^2 b(x; 10, 0.2) = 0.87 - 0.67 = 0.20$$

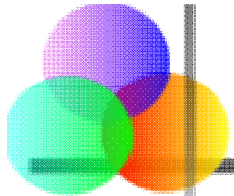
On the other hand, the exact probability is $h(3; 5000, 10, 1000) = 0.2015$



Geometric Distribution

If repeated independent trials can result in a success with probability p and a failure with probability $q = 1 - p$, then the probability distribution of the random variable X , the number of the trial on which the first success occurs, is

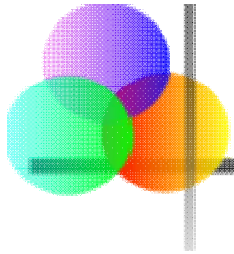
$$g(x; p) = pq^{x-1}, x = 1, 2, 3, \dots$$



Example 5.18: For a certain manufacturing process, it is known that, on the average, 1 in every 100 items is defective. What is the probability that the fifth item inspected is the first defective item found?

Solution: Using the geometric distribution with $x=5$ and $p=0.01$, we have

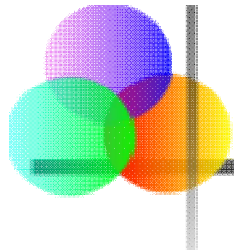
$$g(5;0.01) = (0.01)(0.99)^4 = 0.0096$$



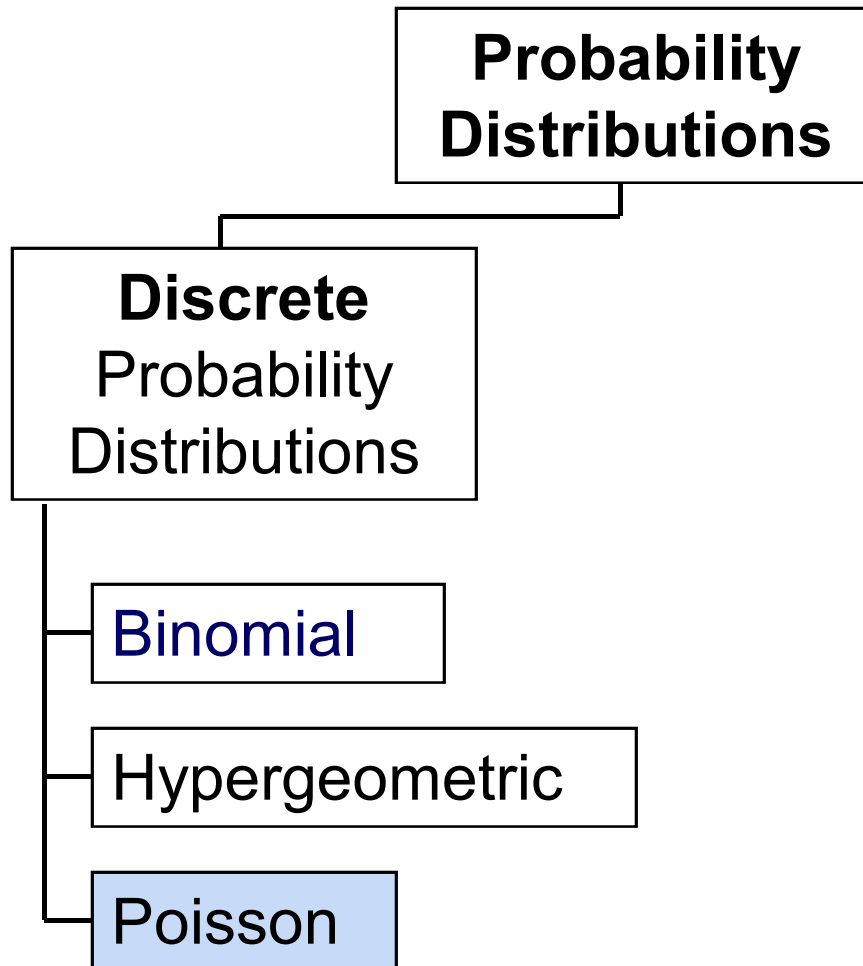
Example 5.19: At a “busy time,” a telephone exchange is very near capacity, so callers have difficulty placing their calls. It may be of interest to know the number of attempts necessary in order to make a connection. Suppose that we let $p=0.05$ be the probability of a connection during a busy time. We are interested in knowing the probability that 5 attempts are necessary for a successful call.

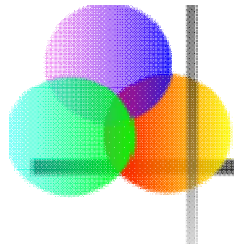
Solution: Using the geometric distribution with $x=5$ and $p=0.05$ yields

$$P(X = x) = g(5;0.05) = (0.05)(0.95)^4 = 0.041$$



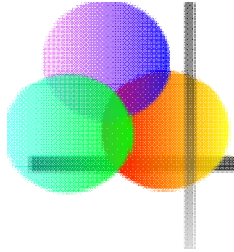
The Poisson Distribution





The Poisson Distribution

- Apply the Poisson Distribution when:
 - You wish to count the number of times an event occurs in a given continuous interval
 - The probability that an event occurs in one subinterval is very small and is the same for all subintervals
 - The number of events that occur in one subinterval is independent of the number of events that occur in the other subintervals
 - There can be no more than one occurrence in each subinterval
 - The average number of events per unit is λ (lambda)



Poisson Distribution Formula

$$P(x, \lambda t) = \frac{e^{-\lambda t} (\lambda t)^x}{x!}, x = 0, 1, 2, \dots$$

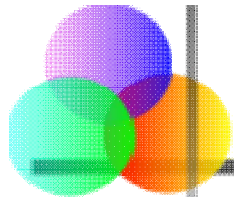
where:

x = number of outcomes in a given time of interval

λ = average number of outcomes per unit of time, distance, area or volume

e = base of the natural logarithm system (2.71828...)

t = time, distance, area or volume of interest.

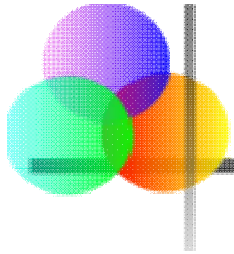


Using Poisson Tables

Table A.2 Poisson Probability Sums $\sum_{x=0}^r p(x; \mu)$

<i>r</i>	μ								
	1.0	1.5	2.0	2.5	3.0	3.5	4.0	4.5	5.0
5	0.9994	0.9955	0.9834	0.9580	0.9161	0.8576	0.7851	0.7029	0.6160
6	0.9999	0.9991	0.9955	0.9858	0.9665	0.9347	0.8893	0.8311	0.7622

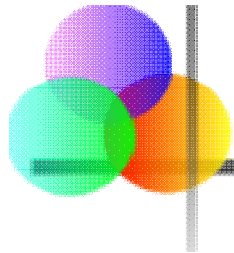
<i>r</i>	μ								
	10.0	11.0	12.0	13.0	14.0	15.0	16.0	17.0	18.0
15	0.9513	0.9074	0.8444	0.7636	0.6694	0.5681	0.4667	0.3715	0.2867



Example 5.20: During a laboratory experiment, the average number of radioactive particles passing through a counter in 1 millisecond is 4. What is the probability that 6 particles enter the counter in a given millisecond?

Solution: Using the Poisson distribution with $x=6$ and $\lambda t= 4$ and referring to Table A.2, we have

$$p(6; 4) = \frac{e^{-4}4^6}{6!} = \sum_{x=0}^6 p(x; 4) - \sum_{x=0}^5 p(x; 4) = 0.8893 - 0.7851 = 0.1042.$$



Example 5.21: Ten is the average number of oil tankers arriving each day at a certain port. The facilities at the port can handle at most 15 tankers per day. What is the probability that on a given day tankers have to be turned away?

Solution: Let X be the number of tankers arriving each day. Then, using Table A.2, we have

$$P(X > 15) = 1 - P(X \leq 15) = 1 - \sum_{x=0}^{15} p(x; 10) = 1 - 0.9513 = 0.0487.$$