BANGLADESH ARMY INTERNATIONAL UNIVERSITY OF SCIENCE AND TECHNOLOGY (BAIUST), CUMILLA

Mid Term Examination, Fall 2024

Department of Computer Science and Engineering (CSE)

Level-1, Term-1

Course Code: CSE 113 **Full Marks:** 90
Course Title: Discrete Mathematics
Credit Hour: 03 **Time:** 1 hr. 30 mins

ANSWER SHEET

Examiner's Note: Answer any three (03) of the following four (04) questions including Q.No.-1.
Student's Note: All four questions are answered below for reference and study purposes.

Question 1

a. What is tautology? Determine whether $(\neg q \land (p \to q)) \to \neg p$ is a tautology.

Definition of Tautology

A proposition is a **tautology** if it is **always true**, regardless of the truth values of the propositional variables that occur in it.

Determining if $(\neg q \land (p \to q)) \to \neg p$ is a Tautology

We can use a truth table to determine if the compound proposition $S \equiv (\neg q \land (p \to q)) \to \neg p$ is a tautology.

p	q	$\neg p$	$\neg q$	$p \to q$	$\neg q \land (p \to q)$	$S \equiv (\neg q \land (p \to q)) \to \neg p$
T	T	F	F	T	F	T
T	F	F	T	F	F	T
F	T	T	F	T	F	T
F	F	T	T	T	T	T

Since the final column for the proposition $(\neg q \land (p \to q)) \to \neg p$ contains only 'T' (True), the proposition is **a tautology**. This is a classical proof known as the **Contrapositive Law** (or *Modus Tollens* written as a tautology).

b. Show that $\neg p \leftrightarrow q$ and $p \leftrightarrow \neg q$ are logically equivalent by developing a series of logical equivalences.

We will start with $\neg p \leftrightarrow q$ and transform it into $p \leftrightarrow \neg q$ using known logical equivalences:

$$\begin{aligned}
\neg p \leftrightarrow q &\equiv (\neg p \to q) \land (q \to \neg p) && \text{(Definition of Biconditional)}\\
&\equiv (\neg(\neg p) \lor q) \land (\neg q \lor \neg p) && \text{(Definition of Implication, } a \to b \equiv \neg a \lor b\text{)}\\
&\equiv (p \lor q) \land (\neg q \lor \neg p) && \text{(Double Negation Law)}\\
&\equiv (\neg(\neg p) \lor \neg q) \land (\neg q \lor \neg p) && \text{(Double Negation on first disjunction, for next step)}\\
&\equiv (\neg p \to \neg q) \land (\neg(\neg q) \to \neg p) && \text{(Definition of Implication, } a \to b \equiv \neg a \lor b\text{)}\\
&\equiv (\neg p \to \neg q) \land (q \to \neg p) && \text{(Double Negation Law)}
\end{aligned}$$

The above derivation shows that $\neg p \leftrightarrow q \equiv (\neg p \to q) \land (q \to \neg p)$.

Let's transform $\neg p \leftrightarrow q$ into the equivalent form of $p \leftrightarrow \neg q$, which is $(p \to \neg q) \land (\neg q \to p)$.

$$\begin{aligned}
\neg p \leftrightarrow q &\equiv (\neg p \to q) \land (q \to \neg p) && \text{(Definition of Biconditional)}\\
&\equiv (\neg q \to \neg(\neg p)) \land (\neg(\neg p) \to \neg q) && \text{(Contrapositive Law, } a \to b \equiv \neg b \to \neg a\text{)}\\
&\equiv (\neg q \to p) \land (p \to \neg q) && \text{(Double Negation Law)}\\
&\equiv (p \to \neg q) \land (\neg q \to p) && \text{(Commutative Law for } \land\text{)}\\
&\equiv p \leftrightarrow \neg q && \text{(Definition of Biconditional)}
\end{aligned}$$

Thus, $\neg p \leftrightarrow q$ is **logically equivalent** to $p \leftrightarrow \neg q$.

c. Write the following propositions using p, q, and logical connectives.

Let p: It is below freezing.
Let q: It is snowing.

1. It is below freezing and snowing.

$$p \land q$$

2. It is below freezing but not snowing. ("but" is logically equivalent to "and")

$$p \land \neg q$$

3. It is not below freezing and it is not snowing.

$$\neg p \land \neg q$$

4. It is either snowing or below freezing (or both). (”or” in this context is inclusive)

$$q \vee p \quad \text{or equivalently } p \vee q$$

Question 2

a. Construct a truth table for each of the following compound propositions

i. $(p \leftrightarrow q) \vee (\neg p \leftrightarrow \neg r)$

p	q	r	$\neg p$	$\neg r$	$p \leftrightarrow q$	$\neg p \leftrightarrow \neg r$	$(p \leftrightarrow q) \vee (\neg p \leftrightarrow \neg r)$
T	T	T	F	F	T	T	T
T	T	F	F	T	T	F	T
T	F	T	F	F	F	T	T
T	F	F	F	T	F	F	F
F	T	T	T	F	F	F	F
F	T	F	T	T	F	T	T
F	F	T	T	F	T	F	T
F	F	F	T	T	T	T	T

ii. $p \rightarrow (\neg q \wedge r)$

p	q	r	$\neg q$	$\neg q \wedge r$	$p \rightarrow (\neg q \wedge r)$
T	T	T	F	F	F
T	T	F	F	F	F
T	F	T	T	T	T
T	F	F	T	F	F
F	T	T	F	F	T
F	T	F	F	F	T
F	F	T	T	T	T
F	F	F	T	F	T

iii. $(p \leftrightarrow q) \oplus (p \leftrightarrow \neg q)$

p	q	$\neg q$	$p \leftrightarrow q$	$p \leftrightarrow \neg q$	$(p \leftrightarrow q) \oplus (p \leftrightarrow \neg q)$
T	T	F	T	F	T
T	F	T	F	T	T
F	T	F	F	T	T
F	F	T	T	F	T

b. What is the power set of the empty set, set $\{\}$ and set $\{0, 1, 2\}$?

The **power set** $\mathcal{P}(A)$ of a set A is the set of all subsets of A.

1. **Power set of the empty set ($\emptyset$):**

$$\mathcal{P}(\emptyset) = \{\emptyset\}$$

2. **Power set of set $\{\}$ (which is the empty set):**

$$\mathcal{P}(\{\}) = \{\{\}\}$$

3. **Power set of set $\{0, 1, 2\}$:**

$$\begin{aligned}\mathcal{P}(\{0,1,2\}) = \{&\emptyset,\\ &\{0\},\{1\},\{2\},\\ &\{0,1\},\{0,2\},\{1,2\},\\ &\{0,1,2\}\ \}\end{aligned}$$

c. Show that $A \times B \neq B \times A$ when A and B are nonempty, unless $A = B$.

Let A and B be two nonempty sets. The Cartesian product $A \times B$ is the set of all ordered pairs (a, b) where $a \in A$ and $b \in B$. The Cartesian product $B \times A$ is the set of all ordered pairs (b, a) where $b \in B$ and $a \in A$.

The sets $A \times B$ and $B \times A$ are equal **if and only if** for every ordered pair (x, y), the condition $(x \in A \wedge y \in B)$ is equivalent to $(x \in B \wedge y \in A)$.

The ordered pair definition states that $(a, b) = (c, d)$ if and only if $a = c$ and $b = d$.

Proof by Contradiction (Assuming $A \neq B$):

Assume $A \neq B$[cite: 39]. Since A and B are nonempty, there must exist at least one element $a \in A$ such that $a \notin B$, OR there exists an element $b \in B$ such that $b \notin A$.

Let's assume there is an element $a_1 \in A$ such that $a_1 \notin B$. Since B is nonempty, there must be some element $b_1 \in B$.

- Consider the ordered pair (a_1, b_1). Since $a_1 \in A$ and $b_1 \in B$, we have $(a_1, b_1) \in A \times B$.
- For this same pair (a_1, b_1) to be in $B \times A$, we must have $a_1 \in B$ and $b_1 \in A$.
- However, we chose a_1 such that $a_1 \notin B$.
- Since $a_1 \notin B$, the pair (a_1, b_1) is an element of $A \times B$ but **not** an element of $B \times A$.

Since $A \times B$ contains an element not present in $B \times A$, we must conclude that $A \times B \neq B \times A$.

Conclusion

The only case where $A \times B = B \times A$ is when the condition $x \in A \iff x \in B$ holds for all elements x, which means $A = B$. Thus, if A and B are nonempty, $A \times B \neq B \times A$ unless $A = B$.

Question 3

a. Let $f(x) = x^2 + 1$ and $g(x) = x + 2$. Determine $f \circ g$ and $g \circ f$.

i. Determine $f \circ g$

The composition $f \circ g$ is defined as $f(g(x))$. We substitute the expression for $g(x)$ into $f(x)$.

$$\begin{aligned} f \circ g(x) &= f(g(x)) \\ &= f(x+2) \\ &= (x+2)^2 + 1 \\ &= (x^2 + 4x + 4) + 1 \\ &= x^2 + 4x + 5 \end{aligned}$$

Therefore, $f \circ g(x) = x^2 + 4x + 5$.

ii. Determine $g \circ f$

The composition $g \circ f$ is defined as $g(f(x))$. We substitute the expression for $f(x)$ into $g(x)$.

$$\begin{aligned} g \circ f(x) &= g(f(x)) \\ &= g(x^2 + 1) \\ &= (x^2 + 1) + 2 \\ &= x^2 + 3 \end{aligned}$$

Therefore, $g \circ f(x) = x^2 + 3$.

b. Find formulae for the sequences with the following first five terms:

Let a_n denote the n-th term of the sequence, where $n \geq 1$.

i. $1, 1/2, 1/4, 1/8, 1/16, \ldots$

This is a geometric sequence where the first term $a_1 = 1$ and the common ratio $r = 1/2$. The n-th term is given by the formula $a_n = a_1 \cdot r^{n-1}$.

$$a_n = 1 \cdot \left(\frac{1}{2}\right)^{n-1} = \frac{1}{2^{n-1}} \quad \text{for } n \geq 1$$

ii. $1, 3, 5, 7, 9, \ldots$

This is an arithmetic sequence where the first term $a_1 = 1$ and the common difference $d = 2$. The n-th term is given by the formula $a_n = a_1 + (n-1)d$.

$$a_n = 1 + (n-1)2 = 1 + 2n - 2 = 2n - 1 \quad \text{for } n \geq 1$$

iii. $1, -1, 1, -1, 1, \ldots$

This is an alternating sequence. The absolute value of the term is 1, and the sign alternates starting with positive. The n-th term can be expressed using a power of -1.

$$a_n = (-1)^{n+1} \quad \text{or } a_n = (-1)^{n-1} \quad \text{for } n \geq 1$$

c. Compute the following sum: $\sum_{j=0}^{8}(2^{j+1} - 2^j)$. What is the value of $\sum_{p\in\{1,2,5\}} p^3$?

i. Compute $\sum_{j=0}^{8}(2^{j+1} - 2^j)$

This sum is a **telescoping sum**.

$$\begin{aligned}\sum_{j=0}^{8}(2^{j+1} - 2^j) &= (2^{0+1} - 2^0) + (2^{1+1} - 2^1) + (2^{2+1} - 2^2) + \cdots + (2^{8+1} - 2^8) \\ &= (2^1 - 2^0) + (2^2 - 2^1) + (2^3 - 2^2) + \cdots + (2^9 - 2^8)\end{aligned}$$

When the terms are expanded, all intermediate terms cancel out:

$$\sum_{j=0}^{8}(2^{j+1} - 2^j) = -2^0 + (2^1 - 2^1) + (2^2 - 2^2) + \cdots + (2^8 - 2^8) + 2^9$$

The sum simplifies to the last term minus the first term:

$$\sum_{j=0}^{8}(2^{j+1} - 2^j) = 2^9 - 2^0 = 512 - 1 = 511$$

Alternatively, by factoring 2^j from the general term:

$$2^{j+1} - 2^j = 2^j(2-1) = 2^j$$

Thus, the sum is a geometric series:

$$\sum_{j=0}^{8} 2^j = 2^0 + 2^1 + \cdots + 2^8$$

Using the formula for the sum of a geometric series $S_n = \frac{a(r^n-1)}{r-1}$, where $a = 1$, $r = 2$, and $n = 9$ (since j goes from 0 to 8):

$$\sum_{j=0}^{8} 2^j = \frac{1(2^9 - 1)}{2 - 1} = \frac{512 - 1}{1} = 511$$

ii. What is the value of $\sum_{p\in\{1,2,5\}} p^3$?

This sum requires summing the cube of each element p in the set $\{1, 2, 5\}$.

$$\sum_{p\in\{1,2,5\}} p^3 = 1^3 + 2^3 + 5^3$$
$$= 1 + 8 + 125$$
$$= 134$$

Question 4 (Mandatory)

a. List all the steps used by Bubble sort to arrange the list $12, 9, 2, 7, 10, 5$ in ascending order.

Bubble sort works by repeatedly stepping through the list, comparing adjacent elements, and swapping them if they are in the wrong order. The pass through the list is repeated until the list is sorted. The list has $n = 6$ elements.

Initial List:

$L = [12, 9, 2, 7, 10, 5]$

Pass 1 (Largest element bubbles to the end)

- Compare $(12, 9)$. Swap. $\rightarrow [9, 12, 2, 7, 10, 5]$
- Compare $(12, 2)$. Swap. $\rightarrow [9, 2, 12, 7, 10, 5]$
- Compare $(12, 7)$. Swap. $\rightarrow [9, 2, 7, 12, 10, 5]$
- Compare $(12, 10)$. Swap. $\rightarrow [9, 2, 7, 10, 12, 5]$
- Compare $(12, 5)$. Swap. $\rightarrow [9, 2, 7, 10, 5, \mathbf{12}]$
  (12 is now in its final position.)

List after Pass 1: $[9, 2, 7, 10, 5, 12]$

Pass 2

- Compare $(9, 2)$. Swap. $\rightarrow [2, 9, 7, 10, 5, 12]$
- Compare $(9, 7)$. Swap. $\rightarrow [2, 7, 9, 10, 5, 12]$
- Compare $(9, 10)$. No swap. $\rightarrow [2, 7, 9, 10, 5, 12]$
- Compare $(10, 5)$. Swap. $\rightarrow [2, 7, 9, 5, 10, 12]$
  (10 is now in its final position among the first five elements.)

List after Pass 2: $[2, 7, 9, 5, 10, 12]$

Pass 3

- Compare $(2, 7)$. No swap. $\rightarrow [2, 7, 9, 5, 10, 12]$
- Compare $(7, 9)$. No swap. $\rightarrow [2, 7, 9, 5, 10, 12]$
- Compare $(9, 5)$. Swap. $\rightarrow [2, 7, 5, 9, 10, 12]$

List after Pass 3: $[2, 7, 5, 9, 10, 12]$

Pass 4

- Compare $(2, 7)$. No swap. $\rightarrow [2, 7, 5, 9, 10, 12]$
- Compare $(7, 5)$. Swap. $\rightarrow [2, 5, 7, 9, 10, 12]$

List after Pass 4: $[2, 5, 7, 9, 10, 12]$

Pass 5

- Compare $(2, 5)$. No swap. $\rightarrow [2, 5, 7, 9, 10, 12]$
- Compare $(5, 7)$. No swap. $\rightarrow [2, 5, 7, 9, 10, 12]$
  (No swaps occurred, indicating the list is sorted.)

Final Sorted List: $[2, 5, 7, 9, 10, 12]$
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b. Give an explicit formula for a function from the set of all integers to the set of positive integers that is onto but is not one-to-one, and also prove it with proper example.

Let $f : \mathbb{Z} \rightarrow \mathbb{Z}^+$ where $\mathbb{Z}$ is the set of all integers ($\{\ldots, -2, -1, 0, 1, 2, \ldots\}$) and $\mathbb{Z}^+$ is the set of positive integers ($\{1, 2, 3, \ldots\}$).

Explicit Formula

We need a function that maps both positive and negative integers (and zero) to positive integers. The absolute value function is a key component.

$$f(x) = |x| + 1$$

Proof of 'Not One-to-One' (Injective)

A function f is not one-to-one if there exist two distinct inputs x_1 and x_2 in the domain such that $f(x_1) = f(x_2)$.

Let $x_1 = 1$ and $x_2 = -1$.

- $f(x_1) = f(1) = |1| + 1 = 1 + 1 = 2$
- $f(x_2) = f(-1) = |-1| + 1 = 1 + 1 = 2$

Since $x_1 \neq x_2$ but $f(x_1) = f(x_2) = 2$, the function $f(x) = |x| + 1$ is **not one-to-one**.

Proof of 'Onto' (Surjective)

A function $f : A \to B$ is onto if for every element y in the codomain B ($\mathbb{Z}^+$), there exists an element x in the domain A ($\mathbb{Z}$) such that $f(x) = y$.

Let y be any arbitrary positive integer ($y \in \{1, 2, 3, \dots\}$). We need to find an integer x such that $f(x) = y$.

$$f(x) = |x| + 1 = y$$

Solving for $|x|$:

$$|x| = y - 1$$

Since $y \in \mathbb{Z}^+$, the smallest value y can take is 1.

- If $y = 1$, then $|x| = 1 - 1 = 0$, which means $x = 0$. Since $0 \in \mathbb{Z}$, $f(0) = 1$.
- If $y > 1$, then $y - 1$ is a positive integer k. We have $|x| = k$. This means x can be either k or $-k$.

Since k and $-k$ are both integers, for any $y \in \mathbb{Z}^+$, we can find $x = y - 1$ (if $y \geq 1$) such that $x \in \mathbb{Z}$ and $f(x) = y$. Specifically, we can choose $x = y - 1$ (which is a non-negative integer) to show the existence of a pre-image. Since every element in the codomain $\mathbb{Z}^+$ has at least one pre-image in the domain $\mathbb{Z}$, the function $f(x) = |x| + 1$ is **onto**.