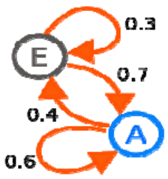
Lecture 15: Introduction to Markov Chains



Stochastic Processes

- Processes that evolve over time in a probabilistic manner are called stochastic processes.
- A stochastic process is defined to be an indexed collection of random variables $\{X_t\}$, where the index t runs through a given set T.
- Often T is taken to be the set of nonnegative integers, and X_t represents a measurable characteristic of interest at time t.
- For example, X_t might represent the inventory level of a particular product at the end of week t.

Stochastic Processes

- A stochastic process often has the following structure:

The current status of the system can fall into any one of $M + 1$ mutually exclusive categories called **states.** For notational convenience, these states are labeled 0, 1, . . . , M. The random variable X_t represents the *state of the system* at time t, so its only possible values are 0, 1, . . . , M. The system is observed at particular points of time, labeled $t = 0, 1, 2, \ldots$. Thus, the stochastic process $\{X_t\} = \{X_0, X_1, X_2, \ldots\}$ provides a mathematical representation of how the status of the physical system evolves over time.

- This kind of process is referred to as being a discrete time stochastic process with a finite state space.

An Inventory Example

- A camera store stocks a particular model camera that can be ordered weekly. Let D_1, D_2, . . . represent the demand for this camera (the number of units that would be sold if the inventory is not depleted) during the first week, second week, . . . , respectively. It is assumed that the D_i are independent and identically distributed random variables having a Poisson distribution with a mean of 1.
- Let X_0 represent the number of cameras on hand at the outset, X_1 the number of cameras on hand at the end of week 1, X_2 the number of cameras on hand at the end of week 2, and so on. Assume that $X_0 = 3$. On Saturday night the store places an order that is delivered in time for the next opening of the store on Monday. The store uses the following order policy:
  - If there are no cameras in stock, the store orders 3 cameras. However, if there are any cameras in stock, no order is placed.

An Inventory Example

- Sales are lost when demand exceeds the inventory on hand.
- Thus, $\{X_t\}$ for $t = 0, 1, \dots$ is a stochastic process of the form just described.
- The possible states of the process are the integers 0, 1, 2, 3, representing the possible number of cameras on hand at the end of the week.
- The random variables X_t are dependent and may be evaluated iteratively by the expression

$$X_{t+1} = \begin{cases} \max\{3 - D_{t+1}, 0\} & \text{if } X_t = 0 \\ \max\{X_t - D_{t+1}, 0\} & \text{if } X_t \geq 1, \end{cases}$$

for $t = 0, 1, 2, \dots$.

Markov Chains

- Markov chain is a special kind of stochastic process with the Markov property.
- Markov property → probabilities involving how the process will evolve in the future depend only on the present state of the process, and so are independent of events in the past.

Markov Chains

- Discrete time Markov chain
- Continuous time Markov chain

Markov Chains

A stochastic process $\{X_t\}$ is said to have the **Markovian property** if $P\{X_{t+1} = j \mid X_0 = k_0, X_1 = k_1, \dots, X_{t-1} = k_{t-1}, X_t = i\} = P\{X_{t+1} = j \mid X_t = i\}$, for $t = 0, 1, \dots$ and every sequence $i, j, k_0, k_1, \dots, k_{t-1}$.

A stochastic process $\{X_t\}$ $(t = 0, 1, \dots)$ is a **Markov chain** if it has the *Markovian property*.

Markov Chains

- The conditional probabilities $P\{X_{t+1} = j \mid X_t = i\}$ for a Markov chain are called (onestep) **transition probabilities.**
- If, for each i and j, $P\{X_{t+1} = j \mid X_t = i\} = P\{X_1 = j \mid X_0 = i\}$, for all t = 1, 2, ……, then the (one-step) transition probabilities are said to be *stationary*.
- Thus, having **stationary transition probabilities** implies that the transition probabilities do not change over time.

Markov Chains

- The existence of stationary (one-step) transition probabilities also implies that, for each $i, j,$ and n (n = 0, 1, 2, . . .), $P\{X_{t+n} = j \mid X_t = i\} = P\{X_n = j \mid X_0 = i\}$, for all t = 0, 1, ...
- These conditional probabilities are called ***n*-step transition probabilities.**
- To simplify notation with stationary transition probabilities, let

$$p_{ij} = P\{X_{t+1} = j \mid X_t = i\},$$
$$p_{ij}^{(n)} = P\{X_{t+n} = j \mid X_t = i\}.$$

Markov Chains

- Because the $p_{ij}^{(n)}$ are conditional probabilities, they must be nonnegative, and since the process must make a transition into some state, they must satisfy the properties

$$p_{ij}^{(n)} \geq 0, \qquad \text{for all } i \text{ and } j;\ n = 0, 1, 2, \ldots,$$

and

$$\sum_{j=0}^{M} p_{ij}^{(n)} = 1 \qquad \text{for all } i;\ n = 0, 1, 2, \ldots.$$

Markov Chains

- A convenient way of showing all the n-step transition probabilities is the matrix form

$$\mathbf{P}^{(n)} = \begin{array}{c|cccc} \text{State} & 0 & 1 & \cdots & M \\ \hline 0 & p_{00}^{(n)} & p_{01}^{(n)} & \cdots & p_{0M}^{(n)} \\ 1 & p_{10}^{(n)} & p_{11}^{(n)} & \cdots & p_{1M}^{(n)} \\ \vdots & \cdots & \cdots & \cdots & \cdots \\ M & p_{M0}^{(n)} & p_{M1}^{(n)} & \cdots & p_{MM}^{(n)} \end{array}, \qquad \text{for } n = 0, 1, 2, \ldots$$

Markov Chains

- Equivalently, the n-step transition matrix

$$\mathbf{P}^{(n)} = \begin{array}{c} \text{State} \\ 0 \\ 1 \\ \vdots \\ M \end{array} \begin{array}{c} \begin{array}{cccc} 0 & 1 & \cdots & M \end{array} \\ \begin{bmatrix} p_{00}^{(n)} & p_{01}^{(n)} & \cdots & p_{0M}^{(n)} \\ p_{10}^{(n)} & p_{11}^{(n)} & \cdots & p_{1M}^{(n)} \\ \cdots & \cdots & \cdots & \cdots \\ p_{M0}^{(n)} & p_{M1}^{(n)} & \cdots & p_{MM}^{(n)} \end{bmatrix} \end{array}$$

Markov Chains

- The Markov chains to be considered in this lecture have the following properties:
  1. **A finite number of states.**
  2. **Stationary transition probabilities.**
- We also will assume that we know the initial probabilities $P\{X_0 = i\}$ for all i.

Formulating the Inventory Example as a Markov Chain

- X_t represents the state of the system at time t.

$$X_{t+1} = \begin{cases} \max\{3 - D_{t+1}, 0\} & \text{if } X_t = 0 \\ \max\{X_t - D_{t+1}, 0\} & \text{if } X_t \geq 1, \end{cases}$$

for $t = 0, 1, 2, \ldots.$

- Given that the current state is $X_t = i$, the expression above indicates that X_{t+1} depends only on D_{t+1} (the demand in week t+1) and X_t.
- Since X_{t+1} is independent of any past history of the inventory system, the stochastic process $\{X_t\}$ ($t = 0, 1, ..$) has the Markovian property and so is a Markov chain.

Formulating the Inventory Example as a Markov Chain

- Now consider how to obtain the (one-step) transition probabilities, i.e., the elements of the (one-step) *transition matrix*

$$\mathbf{P} = \begin{array}{c} \text{State} \\ 0 \\ 1 \\ 2 \\ 3 \end{array} \begin{array}{c} \begin{array}{cccc} 0 & 1 & 2 & 3 \end{array} \\ \begin{bmatrix} p_{00} & p_{01} & p_{02} & p_{03} \\ p_{10} & p_{11} & p_{12} & p_{13} \\ p_{20} & p_{21} & p_{22} & p_{23} \\ p_{30} & p_{31} & p_{32} & p_{33} \end{bmatrix} \end{array}$$

- given that D_{t+1} has a Poisson distribution with a mean of 1.

Formulating the Inventory Example as a Markov Chain

$$P\{D_{t+1} = n\} = \frac{(1)^n e^{-1}}{n!}, \qquad \text{for } n = 0, 1, \ldots,$$

$$P\{D_{t+1} = 0\} = e^{-1} = 0.368,$$
$$P\{D_{t+1} = 1\} = e^{-1} = 0.368,$$
$$P\{D_{t+1} = 2\} = \frac{1}{2}e^{-1} = 0.184,$$
$$P\{D_{t+1} \geq 3\} = 1 - P\{D_{t+1} \leq 2\} = 1 - (0.368 + 0.368 + 0.184) = 0.080.$$

Formulating the Inventory Example as a Markov Chain

- For the first row of **P,** we are dealing with a transition from state $X_t = 0$ to some state X_{t+1}.

$$X_{t+1} = \max\{3 - D_{t+1}, 0\} \qquad \text{if } X_t = 0.$$

Therefore, for the transition to $X_{t+1} = 3$ or $X_{t+1} = 2$ or $X_{t+1} = 1$,

$$p_{03} = P\{D_{t+1} = 0\} = 0.368,$$
$$p_{02} = P\{D_{t+1} = 1\} = 0.368,$$
$$p_{01} = P\{D_{t+1} = 2\} = 0.184.$$

Formulating the Inventory Example as a Markov Chain

- A transition from $X_t = 0$ to $X_{t+1} = 0$ implies that the demand for cameras in week t+1 is 3 or more after 3 cameras are added to the depleted inventory at the beginning of the week, so

$$p_{00} = P\{D_{t+1} \geq 3\} = 0.080.$$

Formulating the Inventory Example as a Markov Chain

- For the other rows of **P,** the formula for the next state is

$$X_{t+1} = \max\{X_t - D_{t+1}, 0\} \qquad \text{if } X_t \geq 1.$$

This implies that $X_{t+1} \leq X_t$, so $p_{12} = 0$, $p_{13} = 0$, and $p_{23} = 0$.

- For the other transitions,

$$p_{11} = P\{D_{t+1} = 0\} = 0.368,$$
$$p_{10} = P\{D_{t+1} \geq 1\} = 1 - P\{D_{t+1} = 0\} = 0.632,$$
$$p_{22} = P\{D_{t+1} = 0\} = 0.368,$$
$$p_{21} = P\{D_{t+1} = 1\} = 0.368,$$
$$p_{20} = P\{D_{t+1} \geq 2\} = 1 - P\{D_{t+1} \leq 1\} = 1 - (0.368 + 0.368) = 0.264.$$

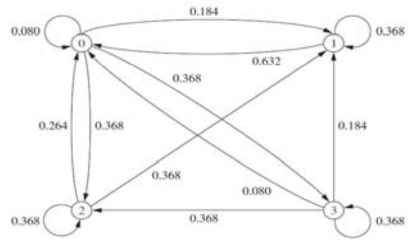
Formulating the Inventory Example as a Markov Chain

- For the last row of **P**, week t+1 begins with 3 cameras in inventory, so the calculations for the transition probabilities are exactly the same as for the first row.
- Consequently, the complete transition matrix is

$$\mathbf{P} = \begin{array}{c} \text{State} \\ 0 \\ 1 \\ 2 \\ 3 \end{array} \begin{array}{c} \begin{array}{cccc} 0 & 1 & 2 & 3 \end{array} \\ \begin{bmatrix} 0.080 & 0.184 & 0.368 & 0.368 \\ 0.632 & 0.368 & 0 & 0 \\ 0.264 & 0.368 & 0.368 & 0 \\ 0.080 & 0.184 & 0.368 & 0.368 \end{bmatrix} \end{array}$$

Formulating the Inventory Example as a Markov Chain

- The information given by this transition matrix can also be depicted graphically with the state transition diagram.



Continuous Time Markov Chains

- When the evolution of the process is being observed continuously over time, the corresponding stochastic process is called continuous process.
- Example: Queuing models.
- As before, we label the possible **states** of the system as 0, 1, . . . , M.
- Starting at time 0 and letting the time parameter t' run continuously for $t' \geq 0$, we let the random variable $X(t')$ be the state of the system at time t'.
- Thus, $X(t')$ will take on one of its possible (M+1) values *over some interval,* $0 \leq t' \leq t_1$, then will jump to another value over the next interval, $t_1 \leq t' \leq t_2$, *etc.,* where these transit points (t_1, t_2, . . .) are random points in time (not necessarily integer).

Continuous Time Markov Chains

- Therefore, the state of the system now has been observed at times $t' = s$ and $t' = r$. *Label* these states as

$$X(s) = i \quad \text{and} \quad X(r) = x(r).$$

Given this information, it now would be natural to seek the probability distribution of the state of the system at time $t' = s + t$. In other words, what is

$$P\{X(s+t) = j \mid X(s) = i \text{ and } X(r) = x(r)\}, \quad \text{for } j = 0, 1, \ldots, M?$$

Continuous Time Markov Chains

A continuous time stochastic process $\{X(t'); t' \geq 0\}$ has the **Markovian property** if
$P\{X(t+s) = j \mid X(s) = i \text{ and } X(r) = x(r)\} = P\{X(t+s) = j \mid X(s) = i\}$,
for all $i, j = 0, 1, \ldots, M$ and for all $r \geq 0$, $s > r$, and $t > 0$.

$P\{X(t+s) = j \mid X(s) = i\}$ is a **transition probability**

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Continuous Time Markov Chains

If the transition probabilities are independent of s, so that
$P\{X(t+s) = j \mid X(s) = i\} = P\{X(t) = j \mid X(0) = i\}$
for all $s > 0$, they are called stationary transition probabilities.

Having stationary transition probabilities implies that the transition probabilities do not change over time.

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